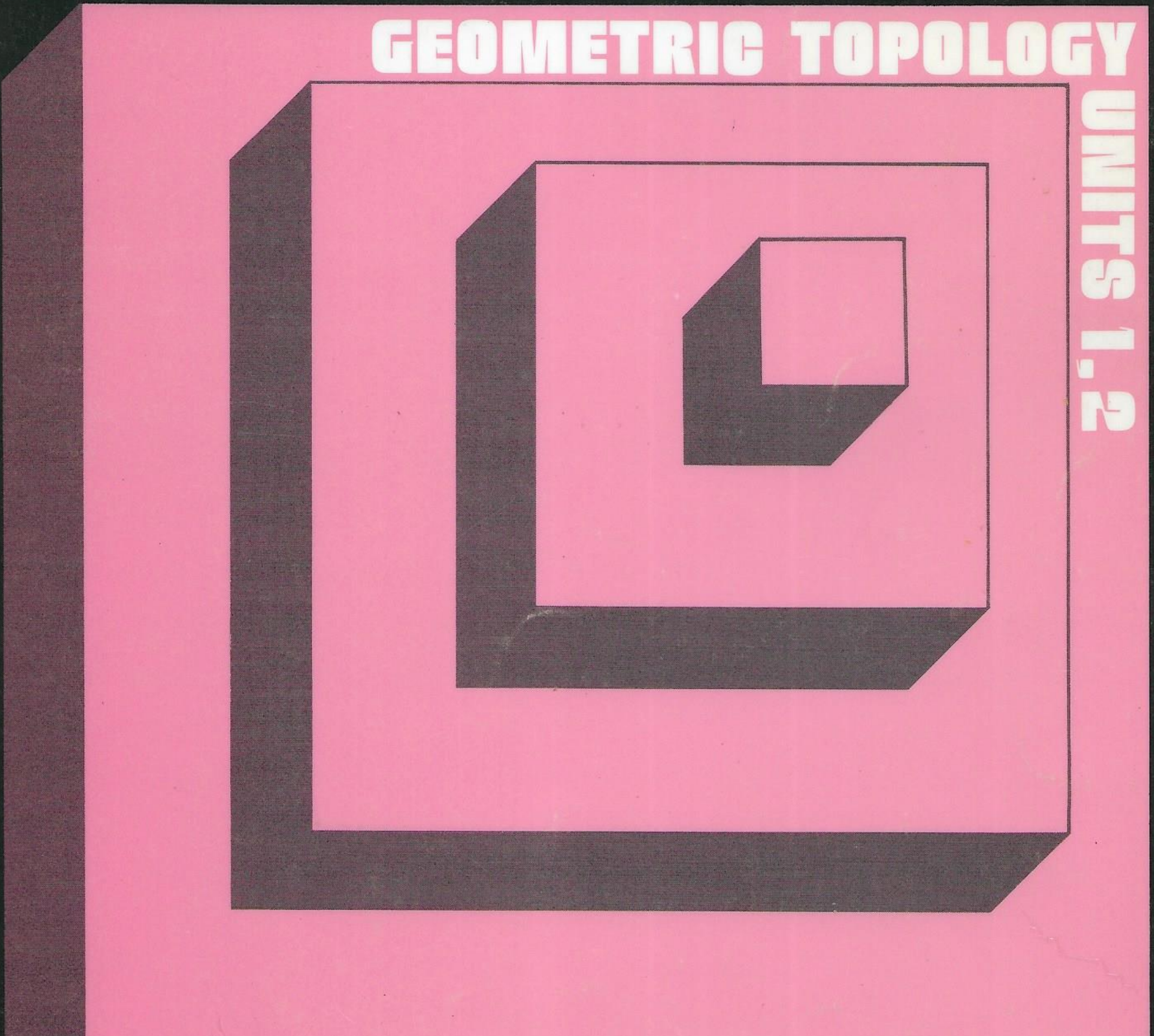
M335 GT 1,2

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

UNITS 1,2



THE OPEN UNIVERSITY
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M335 Studies in Pure Mathematics
Prepared by the Course Team

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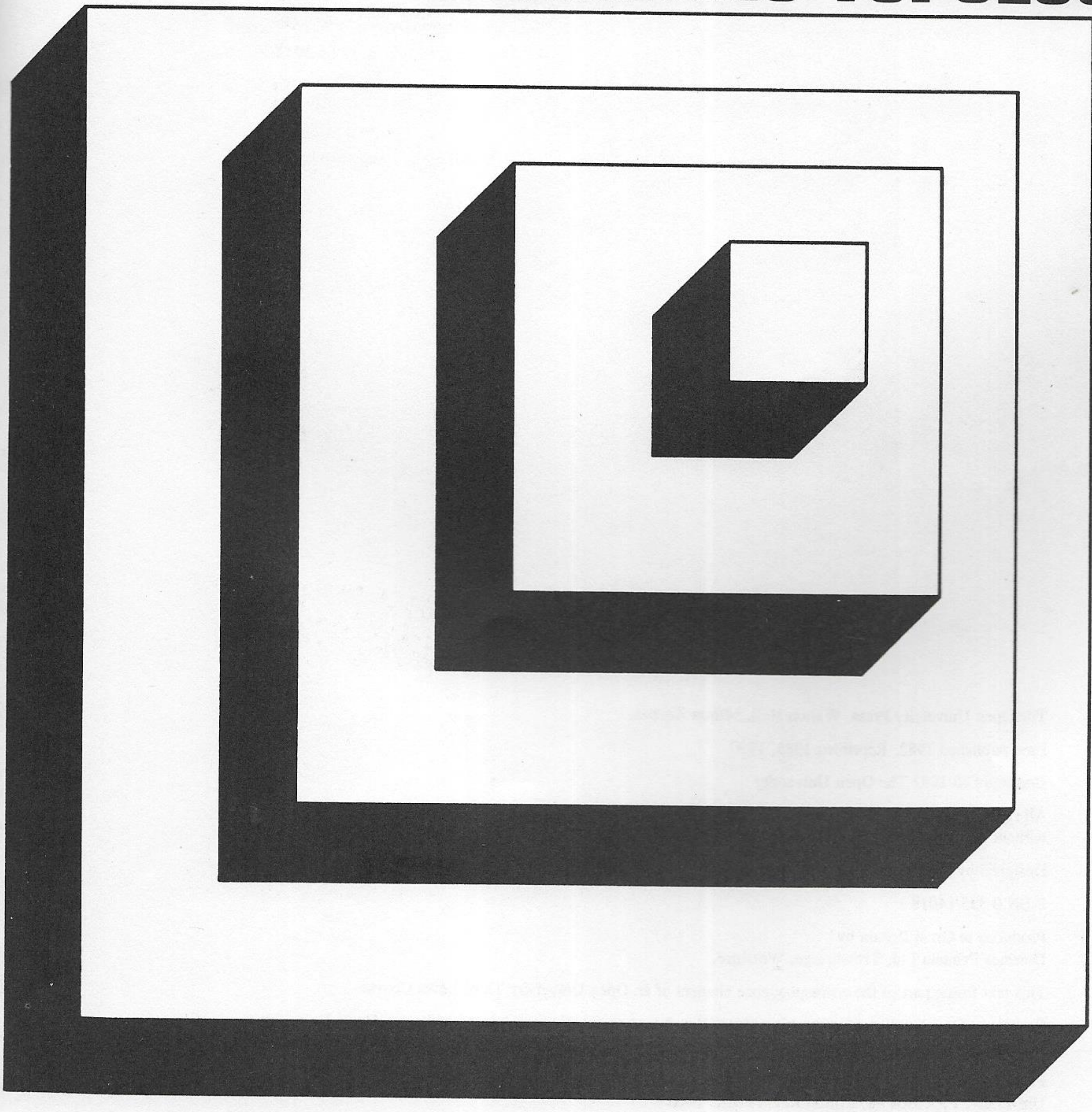

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1.3

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Introduction to the Option

The Overall Course

This course is one of the options for the Open University half-credit course M335: *Studies in Pure Mathematics*. For information about the overall course, including instructions concerning the continuous assessment and final examination, see *Guide to the Course M335*.

The Set Textbook

The set textbook for the course is the 1982 Open University Edition of *Elementary Topology* by D. W. Blackett, published by Academic Press. It is essential that you possess a copy of this edition.

We have not attempted to cover the whole of Blackett's book; this is not because the material is uninteresting, it is simply in order to try to give you a broad introduction to one of the "growth areas" of mathematics in a very limited time. You should feel free to read any sections of Blackett which we omit, provided that this does not cause you to fall behind with the course.

The Course Units

The material has been divided into eight roughly equal parts which we present as "units". Each unit, together with the related television programme and assessment material, should involve you in about 10–12 hours work. Most units have a section of the textbook at heart. Reading passages from Blackett (**B**) are referred to in the following way:

READ B Section 1.1, page 1 to page 6, line 9.

(In this context, line $-n$ means n lines up from the bottom of the page.)

Notes on reading passages are referenced in italic, and the end of a Notes section is marked ▲.

Worked examples end with □, where necessary, and the solutions to self-assessment questions (SAQs) end with ■. Examples and SAQs are numbered in order, starting at 1 in each unit.

Television Programmes

The nature of this subject makes it very difficult to teach using only the written word; accordingly, each unit has a television section which we hope will provide you with valuable additional insight, particularly in visualizing some of the examples.

The television programmes are supplemented by cassette-based pre-TV and post-TV work. You will find a more detailed introduction to this in Section 1.4, *which you should consult in advance of Television Programme 1.*

Cassettes

In addition to the television-related cassettes (A1 and A2) there is a further cassette (B). This gives useful help on certain topics which cannot be dealt with at length in the main body of the course. The material in it is not assessed, and, although you can listen to it at any time, it is particularly relevant to *Units 3* and 5.

Course Structure

The first unit is somewhat untypical. Because you may not have studied topology before, it is essential to describe certain basic concepts and techniques. In particular, you must have a "feel" for the ideas of homeomorphism and continuous functions. The detailed study of these is the subject of another course; for our purposes it is quite sufficient to be able to "spot" when two things are homeomorphic, and part of *Unit 1* is devoted to this end. The only assessable content of *Unit 1* occurs in Section 1.4.

Units 2 to *7* are in standard format, each based on a portion of Blackett. *Unit 8* is slightly different and does not depend directly on Blackett. The reason is that the study of differential equations is closely linked with topology, and we have chosen to expand Blackett's treatment to develop material which should be familiar to you.

Exercise Booklet

If you find that it is difficult to gauge your progress, or that the SAQs are insufficient, then try doing the problems in the Exercise Booklet. This will give you an idea of the level of the assessment and will greatly reinforce your knowledge and confidence.

Course Team

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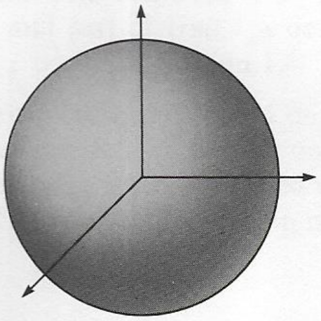
UNIT 1
SURFACES

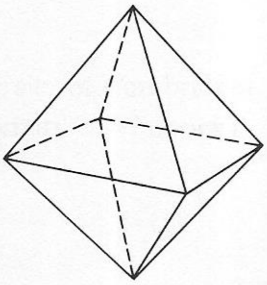
1.1 Surfaces in Euclidean Space

In this course, we shall study surfaces and their properties from a topological standpoint. Which surfaces are familiar to us? One of the easiest to describe is the *sphere*.

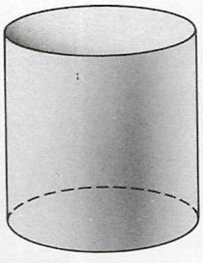


The equation $x^2 + y^2 + z^2 = 1$ defines the unit sphere, centre the origin. In the language of *M101*, the sphere is the *solution set* of the equation, or, as Blackett calls it, the *locus* of the equation. In some ways, describing the sphere as a locus of an equation is inconvenient: it concentrates too much attention on the particular coordinate system, i.e. the position and direction of the coordinate axes. We can recognize a sphere regardless of the coordinate system chosen, and, for that matter, regardless of other details such as the size of the sphere. For this reason, we shall spend a lot of time in this course relying on drawings (or other descriptions) of surfaces. We do not necessarily give equations for surfaces if we can describe them clearly by some other method. This does not mean that equations are not useful, but we can sometimes see better what is happening with the aid of drawings or other geometric aids and then get down to calculations only when the ideas are clearer.

We think in terms of pictures for the moment. What other surfaces shall we be dealing with? The first thing to point out is that surfaces need not be smooth and rounded — they may have sharp edges, like the surface of a regular octahedron.



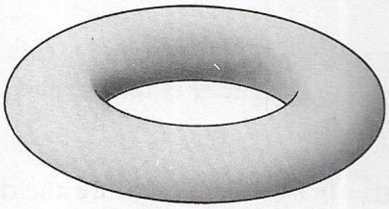
Another way that a surface can differ from the sphere is that it may have a boundary, like the hollow cylinder.



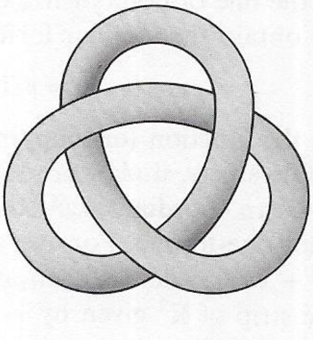
This cylinder has two circles as its boundary. One of the questions we shall consider in Section 1.4 is just what makes the boundary points different from the other points of the cylinder; for the moment we shall rely on just our geometric common sense, or intuition, to justify this. (This use of the word "boundary" is *not* the same as in the Metric and Topological Spaces option of *M335*. Our usage reflects the intuitive idea of an "edge".)

Another important surface is the *torus* — the shape of a doughnut or bicycle inner tube.

This picture is the "standard" torus:

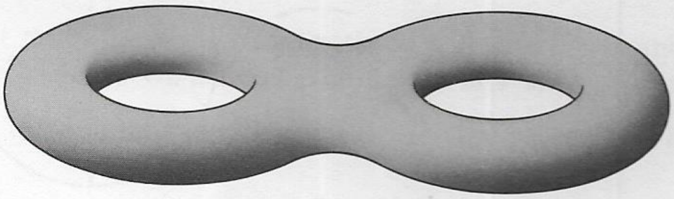


But what about this?

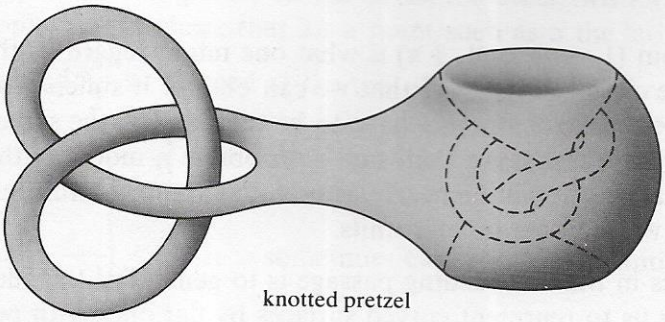


In this course we shall not distinguish between a "standard" and a "knotted" surface. The only difference between the above tori is the way they are "positioned" in $\mathbb{R}^3$. The study of "positioning", or *embeddings* as they are called, is known as knot theory and is an important part of advanced geometric topology. We shall not study this topic, but merely note that surfaces in $\mathbb{R}^3$ can *look* very complicated because of knottedness.

Still more complicated examples are:

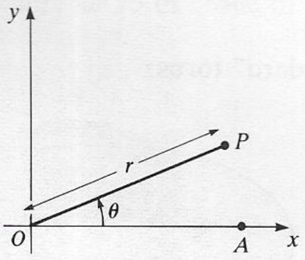


pretzel surface



knotted pretzel

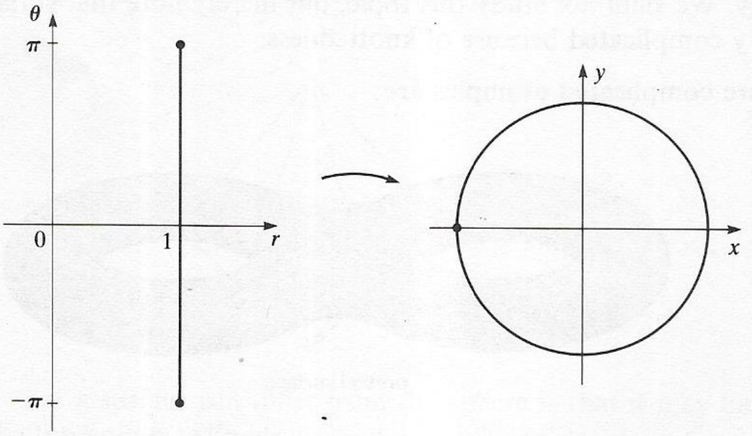
With these examples in mind, we can now get down to some detailed work on representing points on the sphere and torus in various coordinate systems. To see how we can sometimes simplify matters by using a carefully chosen coordinate system, let us look first at a two-dimensional example. Suppose we wish to obtain a convenient representation of points on the circle $x^2 + y^2 = 1$.



We use polar coordinates (r, θ), which measure the distance r of the point P with coordinates (r, θ) from the origin O, and the angle θ which OP makes with a fixed line OA. The equation of the circle becomes $r = 1$. Polar coordinates are conveniently related to Cartesian coordinates by taking x in the direction OA and y in the direction of the line of points with angular coordinate $\theta = \pi/2$. With this convention, we obtain the familiar formulae

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Now suppose we look at the function (or mapping) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $f(r, \theta) = (r\cos\theta,\ r\sin\theta)$. Obviously, if (r, θ) are the polar coordinates of a point, then $f(r, \theta)$ gives the Cartesian coordinates. Also, since $\cos(\theta + 2\pi) = \cos\theta$ and $\sin(\theta + 2\pi) = \sin\theta$, f is not one-to-one. For every (r, θ), $f(r, \theta) = f(r, \theta + 2\pi)$, or, in general, $f(r, \theta) = f(r, \theta + 2n\pi)$, where n is an integer. For this reason, we shall look at f restricted to the strip of $\mathbb{R}^2$ given by $-\pi \leq \theta \leq \pi$. (Both $-\pi$ and π are included — see below.) Now notice which points of the strip are mapped by f into the points of the unit circle $x^2 + y^2 = 1$: we have already seen that these are the points where $r = 1$. Notice that $(1, \pi)$ and $(1, -\pi)$ are both mapped to the same point, $(-1, 0)$.



This line from $(1, -\pi)$ to $(1, +\pi)$ is what one might regard as the best "straight" model of the circle $x^2 + y^2 = 1$ that we can obtain. It suffers from the disadvantage that its endpoints have to be regarded as the same point — stuck together, or *identified* as we shall say — to obtain a model of the circle. But, for many calculations, it will be nearly as useful as the real circle, and easier to describe, as we shall see in later units.

What **B** does in the first reading passage is to generalize this idea to surfaces in $\mathbb{R}^3$, enabling us to represent curved surfaces by flat ones with points "identified". To do this for the sphere and torus we need various coordinate systems in $\mathbb{R}^3$.

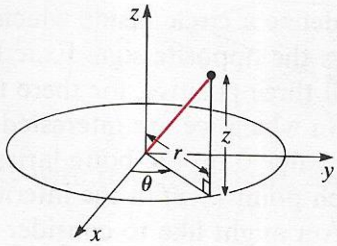
READ B Section 1.1, page 1 to page 6, line 9.

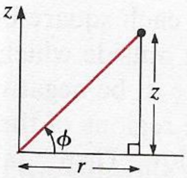
NOTES (i) *Page 4, lines 13–17*

The halfline $\theta = c$. This halfline is the set of all points with polar coordinates (r, c), where we allow r to take any positive value *or zero*, so the origin is included in every halfline of this form. (It is only a convention to include $r = 0$ in each halfline, and sometimes writers *exclude* $r = 0$ from the halfline.) Similarly, in cylindrical coordinates we adopt the convention that the halfplane $\theta = c$ includes the origin, where r and z are both zero.

(ii) *Page 4, line −11*

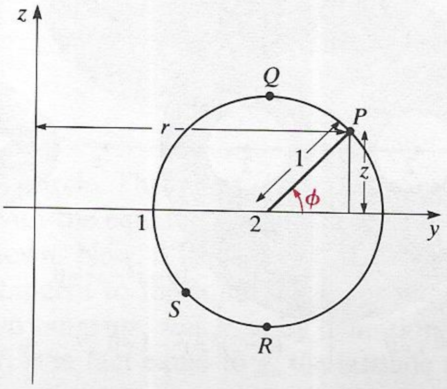
Drawing a picture helps. (You should do this for yourself whenever you find it difficult to visualize what is going on.)





(iii) *Page 5, line −4*

The section of the torus by the plane $x = 0$ is two circles. The right-hand one ($y > 0$) is shown in **B** Figure 1.4.



The point P has Cartesian coordinates (x, y, z) with $x = 0$, and cylindrical coordinates (r, θ, z). We define the latitude ϕ of P as in the diagram, and choose ϕ to be in the range $-\frac{\pi}{2} < \phi \leq \frac{3\pi}{2}$. The points Q, R have $r = 2$ and $z = 1, -1$ respectively; we define the latitudes of Q, R to be $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ respectively. For other points on the circle, the other two formulae on **B** page 5 apply. Let us check that for a point such as S the latitude lies between π and $\frac{3\pi}{2}$ as claimed at the top of **B** page 6. Since $r < 2$,

$\phi = \pi + \arctan\left(\frac{z}{r-2}\right)$ and both z and $r - 2$ are negative. So $\frac{z}{r-2} > 0$ and

$0 < \arctan\left(\frac{z}{r-2}\right) < \frac{\pi}{2}$. (It is sometimes easier to understand a formula like $\phi = \arctan\left(\frac{z}{r-2}\right)$ by first considering the implication

$\tan\phi = \frac{z}{r-2}$.) ▲

EXAMPLE 1 **B** *page 47, Question 1*

Describe the locus of the equation

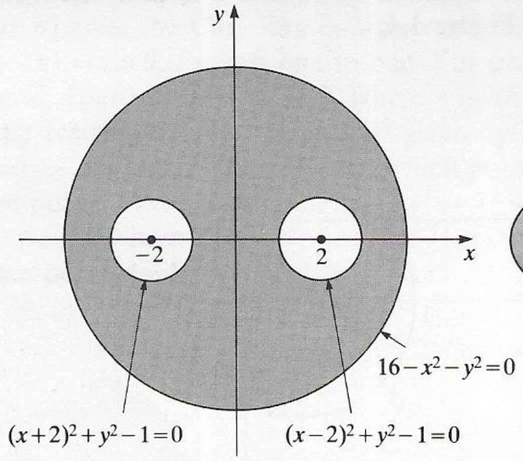
$$z^2 = [(x-2)^2 + y^2 - 1][(x+2)^2 + y^2 - 1][16 - x^2 - y^2]. \qquad (*)$$

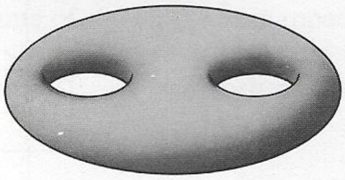
Solution

This kind of advanced sketching can be extremely tricky, but one occasionally has to try it. There are no really systematic methods. A good place to start is to mark out the regions of the *xy*-plane where the right-hand-side of $(*)$ is positive, because it is only over these regions that there are any (real) values of z.

The zeros of each square bracket define a circle inside which the bracket takes one sign and outside which it takes the opposite sign. Exactly two of the brackets have to be negative, or all three positive, for there to be real values of z, and so the regions of the plane in which we are interested are those shaded in the figure below. Then z takes the value 0 on the boundaries of these shaded regions and has two values for each point (x, y) in the interior. An exact sketch of the surface is rather difficult. (You might like to consider this as a challenge problem. As a start, try the cross-section where $x = 0$, i.e. the *yz*-plane.)

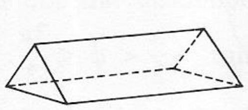
The surface is topologically a pretzel, or two-holed torus.



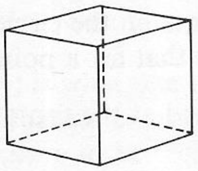


SAQ 1 Sketch the boundaries (if any) of each of the following surfaces in $\mathbb{R}^3$.

(i) A finite hollow prism

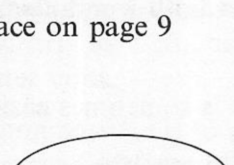


(ii) A cube



(iii) The knotted pretzel surface on page 9

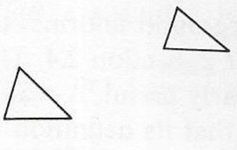
(iv) The Möbius band



SAQ 2 **B** page 47, Question 2.

SOLUTION TO SAQ 1

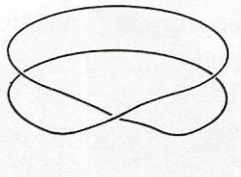
(i) Two triangles



(ii) None

(iii) None

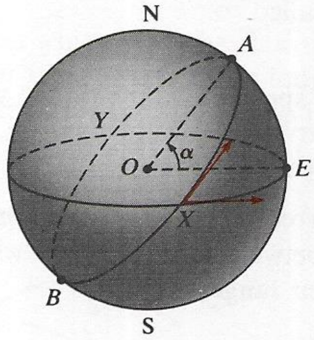
(iv) A circle (twisted!)



■

SOLUTION TO SAQ 2

The main difficulty with this question is visualizing the situation!



Suppose $AXBY$ is the great circle. This great circle is the intersection of the plane making an angle α with the equatorial plane. The point A is the point of maximum latitude α, as shown. Now, it is clear that the tangent to $AXBY$ at X is parallel to AB, and the tangent to the equator at X is parallel to OE. Hence the angle between these two tangents, which is, by definition, the angle at which $AXBY$ crosses the equator, is in fact equal to α, the latitude of A. ■

1.2 Maps of Surfaces

B uses latitude and longitude on a torus to make rectangular maps of a torus, and also uses ordinary latitude and longitude to make rectangular maps of the Earth. These maps are not perfect: apart from distorting shapes (which does not concern us in this course), these maps may have two points which correspond to a single point on the surface. However, we shall see later how this can be used very effectively to investigate other surfaces. **B** concludes the section with stereographic projection — a different kind of map of the sphere (which is used for atlases, though this is not why we are particularly interested in it). The essential difference is that it maps the *whole* of the plane to the sphere in one-to-one fashion, whereas the other maps we have considered map a *bounded* (finite) part of the plane to the sphere.

READ B from page 6, line 10 to the end of Section 1.1.

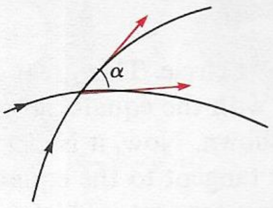
NOTES

(i) *Page 9, line −10*
You will find some background information on maps and stereographic projection in *M203 Unit 2*, Section 2.4. The second television programme of *M203* is also particularly useful. A word of warning: if you do consult *M203*, you should note that its definition of stereographic projection is slightly different. In **B**, projection is onto the equatorial plane, but in *M203* it is on to the parallel plane touching the north pole. (The difference is simply a scale factor of two.)

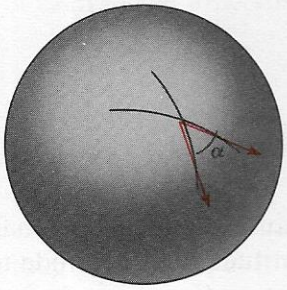
(ii) *Page 11, line 13*
From here to the end of the section, you need not worry about the details. Just remember that stereographic projection is a one-to-one correspondence of the sphere $x^2 + y^2 + z^2 = 1$ with the complex numbers, $\mathbb{C}$, plus the point "∞". The notation for this is $\mathbb{C} \cup \{\infty\}$, and we call $\mathbb{C} \cup \{\infty\}$ the extended plane of complex numbers, or simply the *extended complex plane.* Later, we shall spend some time working with the sphere in this form, and we shall see how to extend the definition of continuous function to apply to $\mathbb{C} \cup \{\infty\}$. For the moment we need only to know what the points of $\mathbb{C} \cup \{\infty\}$ are — they are all the ordinary complex numbers plus the extra point called ∞.

(iii) *Page 11, line −10*
A curve is, intuitively speaking, a line drawn without taking the pencil from the paper (or curved surface of a sphere, etc.). We shall give a more formal definition in Section 1.3.

We have already referred to the angle between two curves in SAQ 2. By definition, the angle between two curves — which have to be smooth — is the angle between their tangents.



This definition is easiest to illustrate in the plane, but it works on the sphere or on any (smooth) surface.

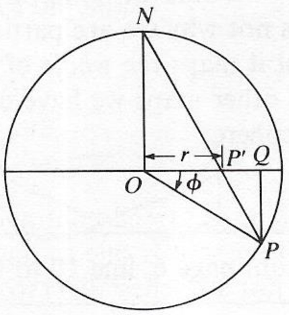


▲

SAQ 3 **B** page 48, Question 7.

SAQ 4 **B** page 47, Question 3.

SOLUTION TO SAQ 3

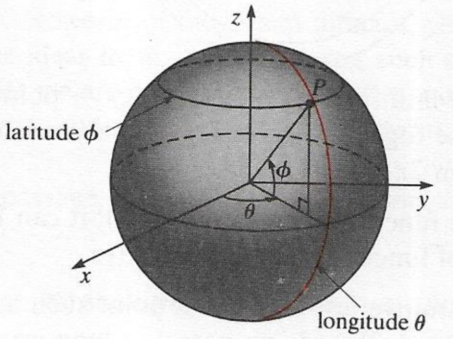


The diagram is labelled to correspond in notation with Figure 1.10 on **B** page 10. The triangles NOP' and PQP' are similar. Hence

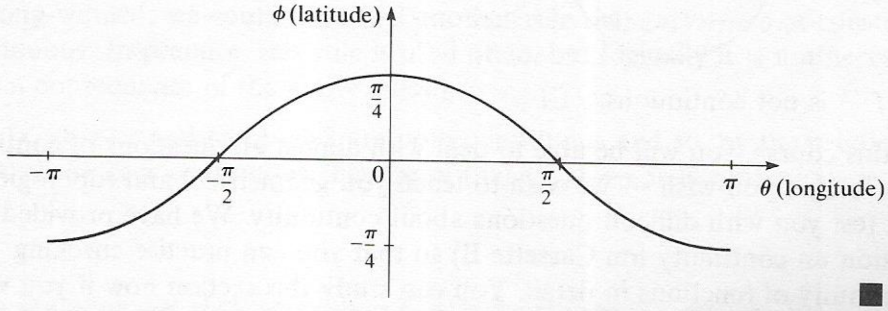
$$\frac{PQ}{NO} = \frac{P'Q}{P'O}, \quad \text{or} \quad \frac{-\sin\phi}{1} = \frac{\cos\phi - r}{r}$$

since ϕ is negative. Now rearrangement gives the formula (1). ■

SOLUTION TO SAQ 4 The point on the sphere with latitude ϕ and longitude θ has (Cartesian) coordinates $(\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$.



Therefore the great circle on which $z = x$ has latitude and longitude which satisfy $\cos\theta\cos\phi = \sin\phi$ or $\tan\phi = \cos\theta$. The longitude can take all values from $-\pi$ to π, and so we can plot the graph of ϕ in terms of θ, which will be the required image. It looks roughly as below, but it is not a cosine curve!



■

1.3 Homeomorphisms for Topological Sphere and Topological Torus

You will be familiar with the concept of continuous functions from (subsets of) $\mathbb{R}$ to $\mathbb{R}$ and from (subsets of) $\mathbb{R}^2$ to $\mathbb{R}$. All definitions of continuous functions seek to make formal the intuitive property that the graph of a continuous function has no sudden breaks in it. Expressing this somewhat loosely, we can say f is continuous at a point c if we can always ensure that $f(x)$ is close to $f(c)$ as x varies on a sufficiently small region containing c.

Although we have provided some additional material on continuous functions, these intuitive ideas are really sufficient for the course.

EXAMPLE 2 To illustrate this, let us look at a function used in **B** (page 14):

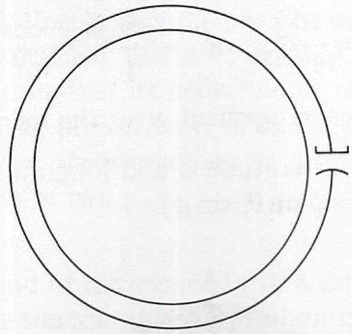
$$f\colon [0,1) \longrightarrow C_1$$

given by $$f(t) = (\cos 2\pi t, \sin 2\pi t),$$

where $[0,1)$ is **B**'s notation for the set of points $\{x : x \in \mathbb{R}, 0 \le x < 1\}$, and C_1 is the unit circle in $\mathbb{R}^2$, i.e. $C_1 = \{(x,y) : x^2 + y^2 = 1, x, y \in \mathbb{R}\}$.

This function is one-to-one, onto and continuous. Properties of the sin and cos functions can be used to prove it is one-to-one and onto, and you can "see" that it is continuous since it just stretches $[0,1)$ to a length 2π — i.e. to $[0,2\pi)$ —

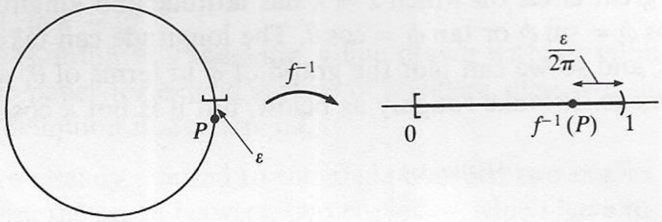
and then "wraps" this interval round the unit circle. (We are not asking you to think of "formal", or "proper", proofs here.)



Clearly, this satisfies the intuitive test of continuity — points that start near to one another end up close together. (In fact, if points start off a distance δ apart, their images are less than $2\pi\delta$ apart in $\mathbb{R}^2$.)

Now look at the inverse function $f^{-1}: C_1 \longrightarrow [0, 1)$; it can be thought of as reversing the sequence of functions illustrated above.

Is it continuous? The intuitive test says no: a point such as P in the fourth quadrant, distance ε from $(1, 0)$, ends up near 1, a long way from 0.



So f^{-1} is not continuous. ☐

In this course, you will be able to deal with almost all questions of continuity at this level, if you wish — we wish to teach you geometrical and topological ideas, not test you with difficult questions about continuity. We have provided a tape section on continuity (on Cassette **B**) so that you can practise checking continuity of functions in detail. You can study this section now if you wish, though we suggest you leave it until after you have studied *Unit 3*. Indeed, if you find you are happy with the approach of **B** as it stands, this tape section could be treated as optional material; it will not be assessed directly. All we expect is that you be able to verify that certain functions that we need to consider are indeed continuous, by applying simple rules to them as explained in *M203 Unit 10*.

Briefly, we need three sets of facts in order to do this.

I Certain simple functions are continuous:

$$\left.\begin{array}{ll}\text{the constant functions} & x \longmapsto c\\ \text{the identity function} & x \longmapsto x\\ \text{the ``absolute value'' function} & x \longmapsto |x|\end{array}\right\}\ \text{are all continuous on any subset of } \mathbb{R}.$$

Certain other functions, such as $\sin x$, $\cos x$ and $\exp x$, are also continuous. These are all functions $X \longrightarrow \mathbb{R}$ where $X \subseteq \mathbb{R}$. For functions on X, where $X \subseteq \mathbb{R}^2$, we also need to know that the *projection* functions are continuous (i.e. the functions $(x, y) \longmapsto x$ and $(x, y) \longmapsto y$ are continuous).

II Rules enable us to deduce that some complicated functions are continuous once we know that certain less complicated ones are continuous. Examples of these rules are:

(a) the sum and product of two continuous functions are continuous;

(b) if f is continuous, so is $\dfrac{1}{f}$, except where $f(x) = 0$;

(c) if f and g are continuous, so is the composite $f \circ g$ (provided the image of g lies in the domain of f).

III We also need a rule which is very useful for functions $F: X \longrightarrow Y$ where X and Y are both subsets of $\mathbb{R}^2$. Suppose that, in coordinate form, $F(x, y) = (f(x, y), g(x, y))$, so that f and g are the "coordinate functions" of F. Then F is continuous if and only if both f and g are continuous functions $X \longrightarrow \mathbb{R}^1$. (This rule generalizes to subsets of higher dimensional space.)

You will find a longer discussion, with examples, in the tape section on Cassette B. If you understand the rules we have just stated, we suggest you do not listen to the tape just now. However, if they seem difficult, you may prefer to listen to the tape or revise the ideas from another source, such as *M203 Unit 10*. Notice that it is just the *rules* we are concerned about here, not the definition of continuity: you can use any definition you are familiar with.

EXAMPLE 3 In the next reading passage, **B** considers the function

$$f(x, y) = (2x, 3y).$$

Using the above rules, we shall prove that this is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

First, we use the result in III to see that f is continuous if and only if the functions $(x, y) \longmapsto 2x$, $(x, y) \longmapsto 3y$ are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}^1$.

By the results in I, $(x, y) \longmapsto x$ is continuous, and so is $x \longmapsto 2$, as a constant function. Hence the composite function $(x, y) \longmapsto 2$ is continuous. (This may seem long-winded: we could have had another rule that $(x, y) \longmapsto c$ (a constant) is continuous. In practice, this rule is used often, but logically it is not necessary, as it is a consequence of the above rules.)

Hence $(x, y) \longmapsto x$ and $(x, y) \longmapsto 2$ are both continuous and so, by the product rule, $(x, y) \longmapsto 2x$ is continuous. The proof that $(x, y) \longmapsto 3y$ is continuous is similar. ☐

Strictly speaking, we shall sometimes need a fourth set of facts which enable us to build up a continuous function on a domain X from continuous functions each defined on only part of X. As an example of this sort of process, let us see how we might prove $x \longmapsto |x|$ is continuous on $\mathbb{R}$ (though this is one of the results in I above). Consider the functions

$$f: x \longmapsto x \quad \text{on } [0, \infty)$$

and

$$g: x \longmapsto -x \quad \text{on } (-\infty, 0].$$

These functions are both continuous on their domains. The function $x \longmapsto |x|$, which we are trying to prove is continuous on $\mathbb{R}$, can be defined by the rule

$$|x| = \begin{cases} f(x), & \text{if } x \in [0, \infty), \\ g(x), & \text{if } x \in (-\infty, 0]. \end{cases}$$

Notice that the two parts of the rule agree where the two domains overlap at $x = 0$. Then the *glue rule* enables us to deduce that $|x|$ is continuous. We shall not need this rule much in the course, though it is discussed in the tape section

The most important continuous functions are *homeomorphisms*. Suppose X, Y are subsets of $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$. Then we say $f: X \longrightarrow Y$ is a homeomorphism if

(a) f is continuous,

(b) f is a bijection (that is, f is both one-to-one and onto), and

(c) the inverse function f^{-1} is continuous (condition (b) guarantees that f^{-1} exists and is a function).

We say "X is homeomorphic to Y" if there is a homeomorphism $f: X \longrightarrow Y$.

You will find a lot of useful background material on homeomorphisms in *M203 Unit 27*. We do not need much of the material explicitly. One result we do need is the fact that the relation "is homeomorphic to" is an equivalence relation. We can therefore speak of the properties which place a set in one equivalence class rather than in another; these properties are called *topological* properties. If X and Y are homeomorphic, they have the same topological properties and are topologically indistinguishable. That is why **B** calls any set homeomorphic to the sphere $x^2 + y^2 + z^2 = 1$ a *topological sphere*, and any set homeomorphic to the standard torus defined in **B** Section 1.1 a *topological torus*, and so on.

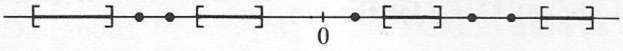
The next reading passage simply summarizes the work on homeomorphisms that will be required later on.

READ B Section 1.2, page 12 to page 14, line -7, and page 15, line -7 to the end of page 16.

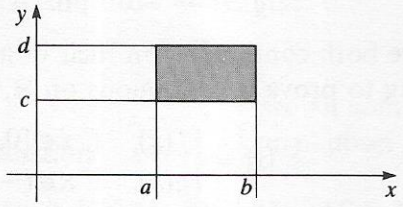
NOTES (i) *Page 13, line 14*
"Connectedness" is defined in *M203 Unit 29*, but the intuitive idea is sufficient for us here. A more useful condition, which implies connectedness, is "path-connected": a surface S is path-connected if, given any two points A, B, there is a path in S from A to B. That is, there is a continuous function $f\colon [0,1] \longrightarrow S$ with $f(0) = A$ and $f(1) = B$ (see *M203 Unit 29*).

(ii) *Page 13, line -9*
This is done as an example in the tape section on continuity.

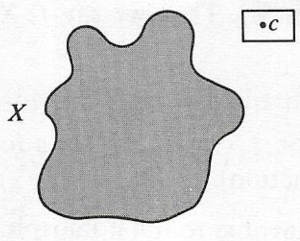
(iii) *Page 14, line -16*
"Closed and bounded". Sets which are both closed and bounded are called *compact*. This definition of compactness is equivalent to the one given in *M203 Unit 29*. "Bounded" is easy to define (and has nothing to do with having a boundary!). A set is bounded if it lies inside a circle of some radius R (probably large). A "closed" set is a generalization of a "closed interval" and is defined in *M203 Unit 27*. In $\mathbb{R}^1$, one closed set is an interval $[a, b]$. A more general closed set in $\mathbb{R}^1$ is a finite number of closed intervals or points (the point a can be thought of as the interval $[a, a]$). There are more complicated closed sets but these do not concern us in this course.



In $\mathbb{R}^2$ a closed rectangle is a set $\{(x, y)\colon a \le x \le b, c \le y \le d\}$.



The important thing to notice is that all the points of the boundary of the rectangle are included. For $X \subseteq \mathbb{R}^2$ to be a closed set, we need to be able to "close off" every point c *not* in X by a closed rectangle, containing c in its *inside* (not on the boundary), which does not meet X.



There is a "simple" theorem in point set topology which is often useful in reducing the amount of work necessary to prove that a particular function is a homeomorphism. We require A, B to be subsets of $\mathbb{R}^n$ for this theorem. The significance of the theorem is that if we know that the function $f: A \longrightarrow B$ (A, B subsets of $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$) satisfies the first two conditions of the homeomorphism definition (page 17), that is, it is a continuous bijection, and, further, that A is closed and bounded (or, as we shall prefer to say, A is a compact set), we can conclude immediately that f is a homeomorphism and need not check explicitly that f^{-1} is also continuous. The truth of this theorem is to be taken on trust, like the continuity rules. ▲

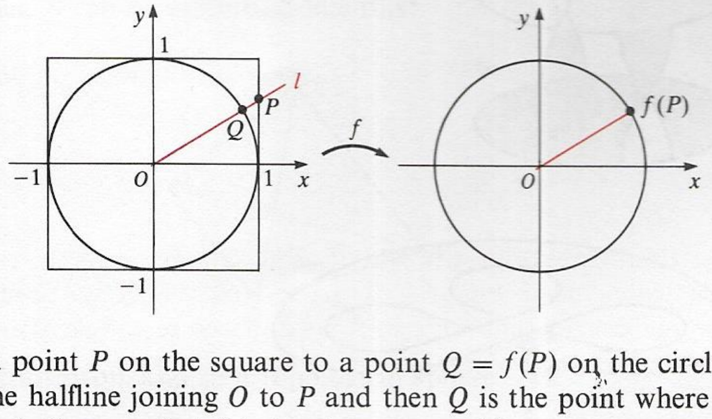
The following Example is provided to fill up a possible gap in the reading passage. You will not need to produce similar proofs in this course.

EXAMPLE 4 Prove that a bijection $f: X \longrightarrow Y$ has an inverse function f^{-1} such that $f^{-1} \circ f$ is the identity function on X and $f \circ f^{-1}$ the identity on Y.

The importance of this example is that while it is almost restating the obvious, it does give us a form of the statement which is convenient for other purposes. For instance, if we wish to prove that the composition of two homeomorphisms $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ is a homeomorphism, we have (amongst other things) to show that $g \circ f$ has a continuous inverse. But we can write immediately $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, because we know f^{-1}, g^{-1} both exist. Since they are continuous, we know their composition is continuous, and hence $(g \circ f)^{-1}$ is too.

First we have to construct the function: it is not too hard to see how. If $y \in Y$, there is *some* $x \in X$ which maps onto y, because f is surjective. We try to define $f^{-1}(y)$ to be this x. The only snag is that there might be another, different, x' which also maps to y, so we should not know whether to choose x or x'. But then $f(x) = y = f(x')$. Since f is also injective, $x = x'$ and the problem cannot arise. So there is a unique x which maps to y and we can indeed define $f^{-1}(y)$ to be this x. So if $f(x) = y$ we can conclude that $f^{-1}(y) = x$. Hence, starting with any x in X we can see that $f^{-1} \circ f(x) = f^{-1}(y) = x$ and hence that $f^{-1} \circ f$ is the identity on X (to use a more formal expression). Similarly, starting from any y in Y, let x in X be the unique element which maps to y under f, so $f(x) = y$. We again have $f^{-1}(y) = x$ so that $f \circ f^{-1}(y) = f(x) = y$ and hence $f \circ f^{-1}$ is the identity on Y.

EXAMPLE 5 Prove that the square and the circle are homeomorphic.



We map a point P on the square to a point $Q = f(P)$ on the circle, as follows. Let l be the halfline joining O to P and then Q is the point where l cuts the circle. We shall prove that this is a homeomorphism.

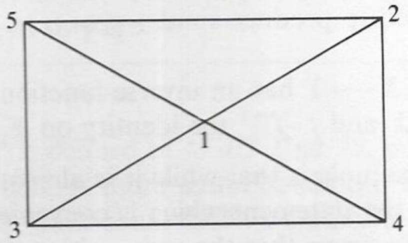
Informally, we can see from the picture that f is a bijection, and also that two nearby points P, P' on the square are mapped by f into two nearby points Q, Q' on the circle. (Q and Q' are at least as close together as P and P'.) Hence f is a continuous bijection. Since the square is *closed* and *bounded*, this proves that f^{-1} is also continuous, so f is a homeomorphism. ☐

We should accept this level of detail in a proof. If you prefer more detail, you can construct a more formal proof along the lines suggested in the tape section. (See also the cone construction in *M203 Unit 27*.)

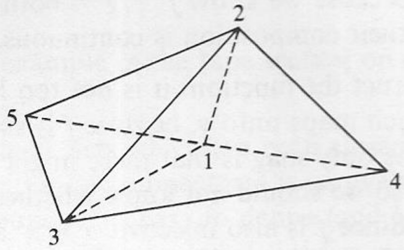
EXAMPLE 6 A surface has vertices 1, 2, 3, 4, 5 and triangular faces 124, 125, 134, 135, 234, 235. Show that it is a topological sphere.

So far we have not seen a systematic method of tackling questions like this. (Such a method will be apparent after you have read **B** Chapter 2.) We might simply draw what we can, and then try to visualize how the rest of the surface can be assembled from suitably shaped triangles.

The first four triangles are simple enough:



Now you may be able to see that 23 has to be outside the plane of the paper. "Fold" the figure along 54, so as to raise 2 and 3 slightly so that we can fit 234 and 235 on top, to give



This is a convex polyhedron, homeomorphic to a sphere.

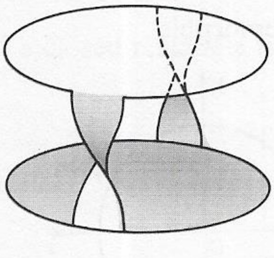
SAQ 5 Which of the following surfaces are (a) path-connected, (b) closed, (c) bounded?

(i) $\{(x, y, 0) : 0 \leq y \leq 1\}$

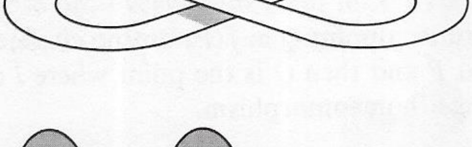
(ii) The punctured disc $\{(x, y, 0) : x^2 + y^2 \leq 1, (x, y) \neq (0, 0)\}$

(iii) The torus

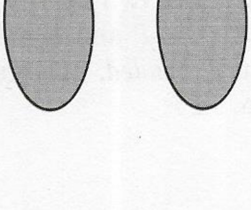
(iv)



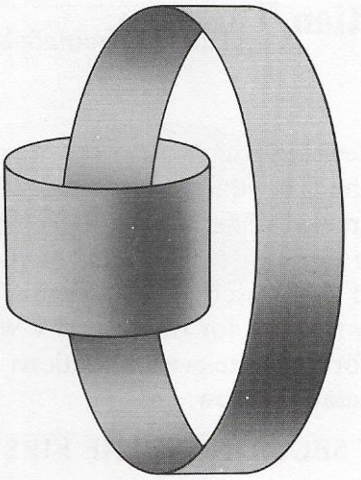
(v)



(vi)



(vii) (Two linked cylinders)



SAQ 6 Prove that the circle C, centre (a,b) and radius r, is homeomorphic to the standard circle S^1, $x^2 + y^2 = 1$. [Hint: Prove that C is homeomorphic to the circle C' with centre $(0,0)$ and radius r, by considering a suitable translation, and then prove C' is homeomorphic to the standard circle.]

SAQ 7 **B** page 48, Question 1.

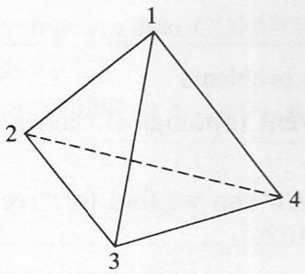
SOLUTION TO SAQ 5 Path-connected: (i), (ii), (iii), (iv), (v).

Closed: (i), (iii), (iv), (v), (vi), (vii).

Bounded: (ii), (iii), (iv), (v), (vi), (vii). ■

SOLUTION TO SAQ 6 The translation $T\colon (x,y) \longmapsto (x-a, y-b)$ is a continuous bijection on the whole of $\mathbb{R}^2$ and so is a continuous bijection on C. Then $T(C) = C'$. Its inverse is also a translation, so its inverse is also continuous. Hence T is a homeomorphism. (Alternatively we could use the theorem described on page 19, since C is closed and bounded.) Now the function $F\colon (x,y) \longmapsto \left(\frac{x}{r}, \frac{y}{r}\right)$ maps the circle C' with equation $x^2 + y^2 = r^2$ onto the circle $x^2 + y^2 = 1$. F is continuous, because $(x,y) \longmapsto \frac{x}{r}$ is continuous and we can use the result in III above. F has an inverse, $(x,y) \longmapsto (rx, ry)$, which is also continuous, and hence F is a homeomorphism. Finally, the composite $F \circ T$ is therefore a homeomorphism between C and S^1. ■

SOLUTION TO SAQ 7 The surface S can be assembled like this:



It is a tetrahedron, and so a topological sphere. ■

1.4 Surfaces, and the Classification Theorem

Introduction

This section, which includes the first television programme and related audio cassette, outlines the main theorem of the first half of the course. It is substantially independent of the first three sections. Before watching the television programme, you should listen to the first "TV-related Audio Cassette" (A1) up to the point where you are told that watching TV1 should be your next activity. You need to allow about thirty minutes for this "pre-TV work", which starts with a short introduction to the format of television sections and continues with a commentary on the material below.

LISTEN TO THE INTRODUCTORY SECTION OF THE FIRST TV-RELATED AUDIO CASSETTE

FRAME 1

Format of TV sections

Print	*Tape*
0. *Introduction* Includes advice about pre-TV work.	*Unit 1* only.
1. *Pre-TV work* Sometimes previous sections of the unit suffice.	Whenever previous sections of the unit do not suffice.

TV PROGRAMME

Print	*Tape*
2. *TV overview*	Every unit.
3. *Post-TV work* (usually problems) (a) For practice; (b) to consolidate and extend the TV material.	Usually not.

Pre-TV Work

Make a note of where the tape for this subsection starts. Then, before restarting the tape, read Frame 2.

FRAME 2

The classification and recognition problems

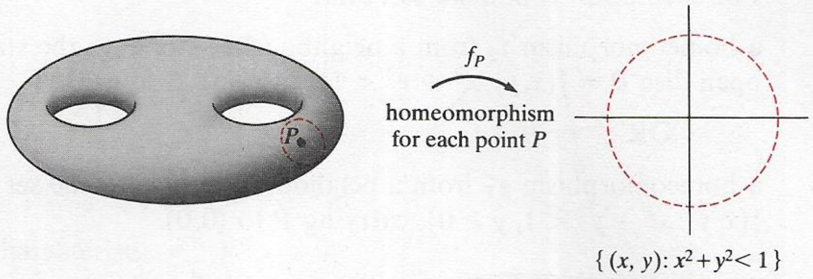
Classification: How many different topological classes of surface are there?

Recognition: Given a surface, how can we find (or "recognize") the topological class that it belongs to?

RESTART THE TAPE

FRAME 3

Rough definition of a surface S



S is "locally disc-like"

FRAME 4

Precise definition of a surface without boundary

[*You are not expected to remember this definition, but read it through.*]

A set S is a surface without boundary if the following conditions hold.

(i) (Frame 3 formalized)
For each point P of S we can find a homeomorphism, f_P, from a subset of S which is a topological neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$, carrying P to $(0, 0)$.
[The inverse homeomorphism is said to "cover P with an open disc". So the whole of S can be "covered by overlapping discs".]

(ii) (The surface is in one piece)
S is *path-connected:* any two of its points can be joined by a path in S.
[A *path* in S is a continuous function $P\colon [0, 1] \longrightarrow S$. This condition simply means that the surface is in one piece — two separate spheres would be disallowed.]

(iii) (Surfaces "live in Euclidean space")
S is homeomorphic to a subset of $\mathbb{R}^n$.
[Therefore S is "metric".]

RESTART THE TAPE

FRAME 5

Surfaces in this course are also COMPACT

i.e. replace (iii) above by:
(iii)′ S is homeomorphic to a closed, bounded ("compact") subset of $\mathbb{R}^n$.

FRAME 6

Surfaces which have a boundary

How should Frame 4(i) be modified?
Do not spend too long on this before referring to Frame 6A.

FRAME 6A

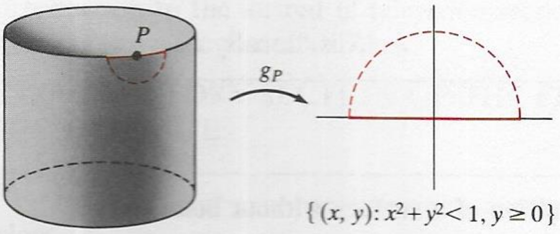
Solution for Frame 6: alternative condition (i) to accommodate boundaries

(i)′ For each point P of S we can find:

a homeomorphism f_P from a neighbourhood of P to the standard open disc $D = \{(x, y): x^2 + y^2 < 1\}$, carrying P to $(0, 0)$.

OR

a homeomorphism g_P from a neighbourhood of P to the set $\{(x, y): x^2 + y^2 < 1, y \geq 0\}$, carrying P to $(0, 0)$.

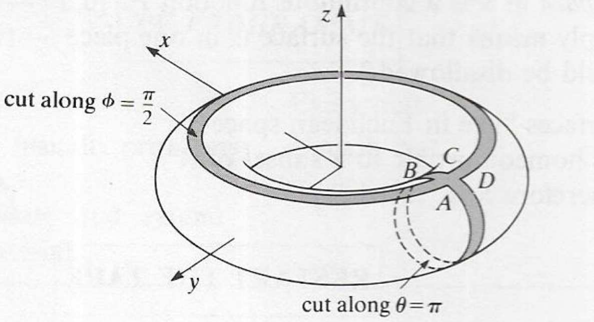

P
g_P
$\{(x, y): x^2+y^2<1, y \geq 0\}$

RESTART THE TAPE

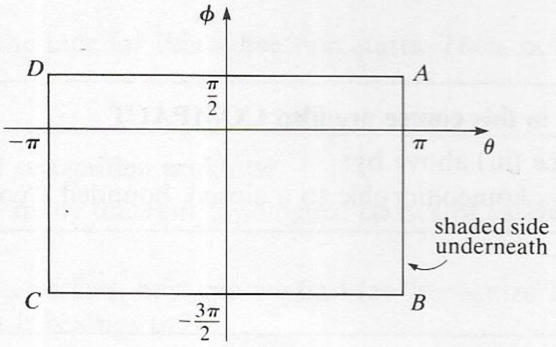
TV Overview

FRAME 7

Cut and flatten a torus


z
x
y
cut along $\phi = \frac{\pi}{2}$
B
C
D
A
cut along $\theta = \pi$

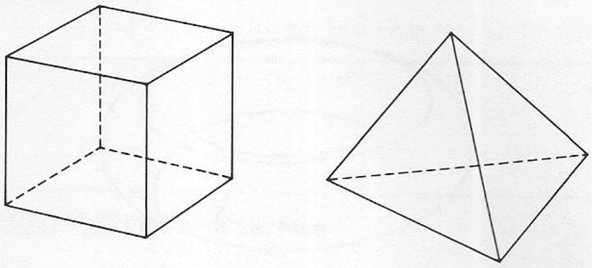
FLATTEN OUT
(inside of torus
facing down)


ϕ
D
A
C
B
$\frac{\pi}{2}$
$-\pi$
π
θ
$-\frac{3\pi}{2}$
shaded side underneath

RESTART THE TAPE

FRAME 8

The cube and tetrahedron



Euler characteristic = $F - E + V$
= faces − edges + vertices

Calculate $F - E + V$ for both polyhedra. Then, before restarting the tape, check your answer with Frame 8A.

FRAME 8A

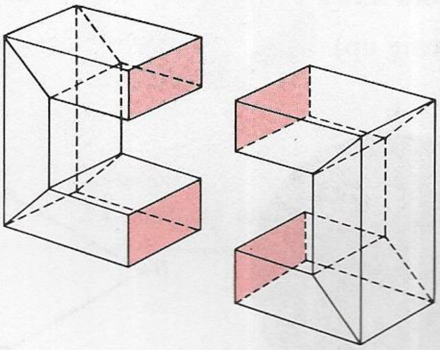
Solution for Frame 8

Euler characteristic of tetrahedron	$F - E + V$ $4 - 6 + 4 = 2$
Euler characteristic of cube	$F - E + V$ $6 - 12 + 8 = 2$

RESTART THE TAPE

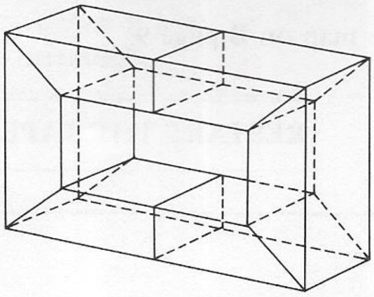
FRAME 9

Two "handle" surfaces



FRAME 10

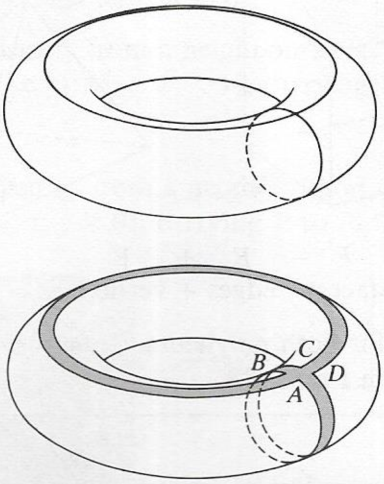
Handle surfaces glued to make a torus

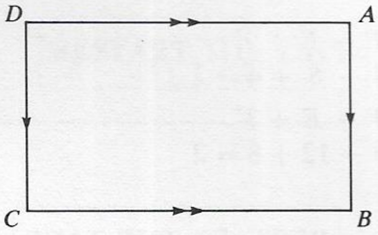


Euler characteristic $F - E + V$ = characteristic of one handle
+ that of the other − 4 (shaded faces)
$= 2 + 2 - 4 = 0$

FRAME 11

Cut and flattened torus (as in Frame 7)



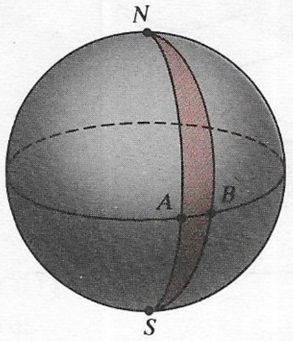


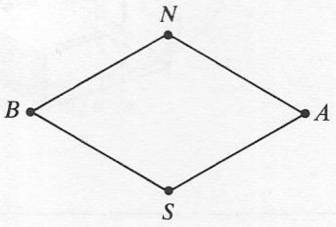
Euler characteristic $F - E + V = 1 - 2 + 1 = 0$

FRAME 12

Sphere cut and flattened into a rhombus

(outside of sphere facing up)





Compare this with the map on **B** page 9.

RESTART THE TAPE

FRAME 13

Euler characteristic calculation for Frame 12

$$F - E + V = 1 - 2 + 3 = 2$$

FRAME 14

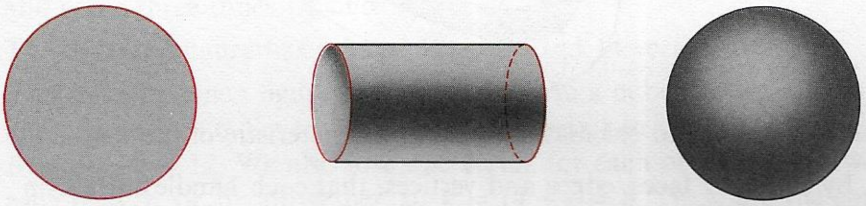
Rules for a SUBDIVISION of a surface

1. Each edge starts or ends at a vertex.
   [It may start and end at the *same* vertex: that is, it may be a loop.]
2. Edges cross or meet only at vertices.
3. The edges form a path-connected network.
4. The faces — which we define to be the separate pieces into which the surface falls when cuts are made along all the edges — are topological discs.

RESTART THE TAPE

FRAME 15

Number of boundary curves is a topological invariant



Disc
(1 boundary curve)

Cylinder
(2 boundary curves)

Sphere
(no boundary curves)

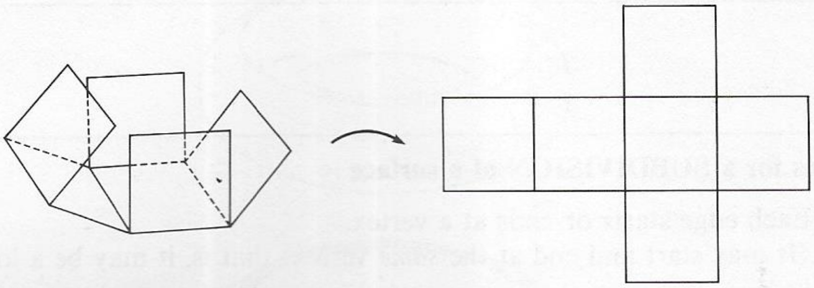
FRAME 16

The Classification Theorem

A surface is determined up to homeomorphism once we know:
its Euler characteristic;
how many boundary curves it has;
whether or not it is orientable.

Post-TV Problems

SAQ 8 The diagram represents a cube that has been cut along seven edges and opened out.



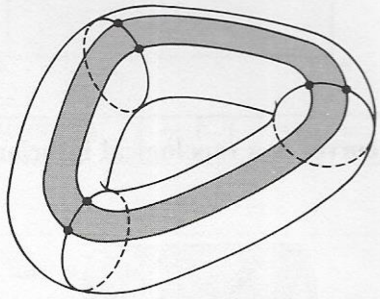
(i) Label the cut edges with letters and arrows to indicate how to reassemble the cube from the flat figure.

(ii) Label the vertices.

(iii) Count the number of edges and vertices in the flat figure.

(iv) Verify that the Euler characteristic can be calculated correctly from the cut and flattened form of the cube, provided that the identifications are taken into account.

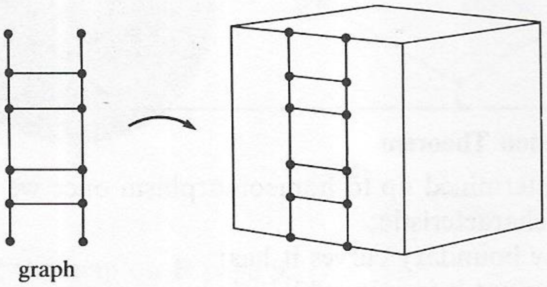
SAQ 9 In the television programme, a subdivision of the torus into six rectangles was shown.



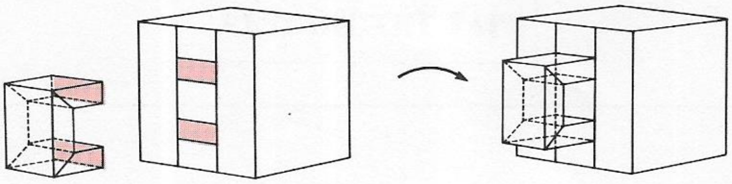
Use this subdivision to calculate the Euler characteristic of the torus.

SAQ 10 Verify, by counting faces, edges and vertices, that each handle surface in Frame 9 has Euler characteristic 2.

SAQ 11 Verify that drawing the graph below on one face of a cube does not alter the Euler characteristic of the cube.



SAQ 12



Show that adding a "handle" surface to a cube with one face marked as in SAQ 11 produces a surface with Euler characteristic 0: an attaché case! Deduce that adding *n* "handle" surfaces to a cube gives a surface with Euler characteristic $2 - 2n$.

SAQ 13 Cut a cube along all its edges so that it becomes six squares.

(i) How many edges, faces and vertices do these squares have?

(ii) How many faces of the original cube share the same edge?

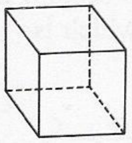
(iii) How many faces of the original cube contain a given vertex?

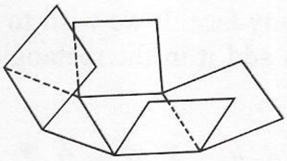
(iv) Use this information to calculate the Euler characteristic of the cube from what you know about the six squares.

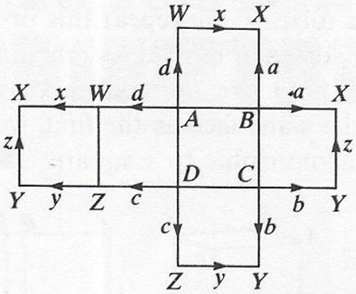
(v) Repeat with the four triangles obtained by cutting a regular tetrahedron along all its edges, and with the eight triangles obtained in a similar way from a regular octahedron.

SOLUTION TO SAQ 8

(i) The seven cut edges are labelled *a*, *b*, *c*, *d*, *x*, *y*, *z* and each edge occurs twice in the flattened figure. There are also five edges which are not labelled because they were not cut.







(ii) There are four vertices (the four vertices which were originally at the top) which are affected by the cuts. After the cut edges are re-glued, all the vertices labelled X come together, so do those labelled Y, and so on. Notice that some vertices appear twice on the flattened figure (Z and W) and some three times (X and Y).

(iii) The flattened figure has 6 faces, 19 edges and 14 vertices.

(iv) When the flattened figure is re-glued to form a cube, the six faces are not affected, so $F = 6$. Seven of the edges disappear in the gluing, so $E = 19 - 7 = 12$. We also lose one vertex for each of Z, W and two for each of X, Y, or six altogether, so $V = 14 - 6 = 8$. These are the correct numbers for the original cube, so we obtain the correct Euler characteristic $6 - 12 + 8 = 2$. ■

SOLUTION TO SAQ 9

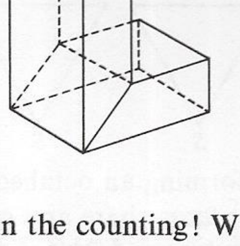
Faces: 3 shaded (at the front) plus 3 unshaded.

Edges: 1 inner triangle (giving 3), 1 outer triangle (another 3), 2 for each of the three hoops; total 12.

Vertices: 6.

$F - E + V = 6 - 12 + 6 = 0$ ■

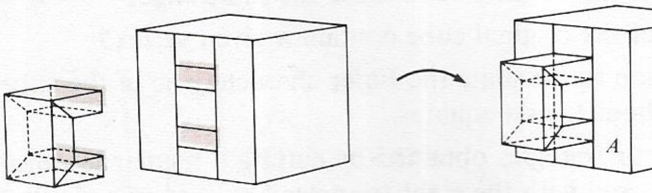
SOLUTION TO SAQ 10



It is easy to make a mistake in the counting! We make it

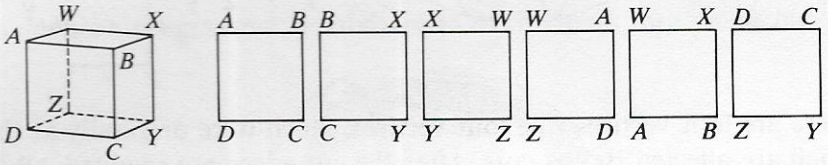
$$F = 14, \quad E = 28, \quad V = 16. \ ■$$

SOLUTION TO SAQ 11 The ladder-like graph in the centre of the face has 12 vertices, and it creates 18 new edges and 6 new faces. Thus its contribution to $F - E + V$ is zero. ■



SOLUTION TO SAQ 12 The count of Faces − Edges + Vertices for the cube and the "handle" surface is, in each case, 2. If we join them, by deleting the shaded faces, then the edges and vertices of these faces (4 of each) make no contribution to the count. Sticking the edges together we have a total count of $2 + 2$ less the four deleted faces, or 0. The resulting "attaché case" (cube with a handle) is, of course, homeomorphic to a torus. If we repeat this process, each time we add a handle we add a count of 2 for each handle less the four deleted faces: so we reduce the count by 2 each time. We can add handles to any face. If we wish to add a second handle to the same face as the first, we can add it in the rectangle A, which is homeomorphic to a square. ■

SOLUTION TO SAQ 13

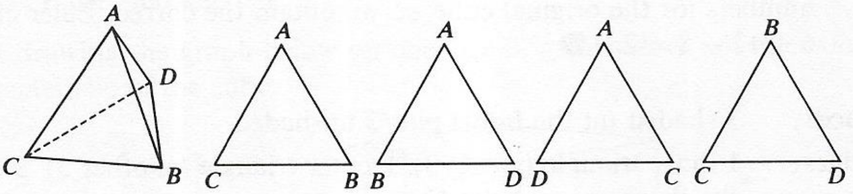


(i) One square has 1 face, 4 edges and 4 vertices: thus six squares have 6 faces, 24 edges and 24 vertices.

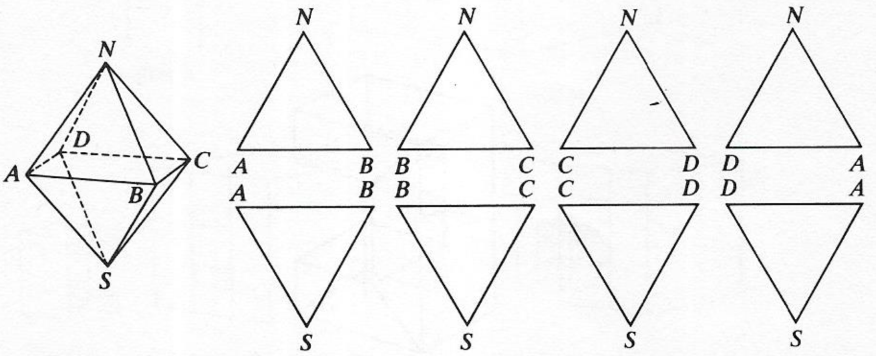
(ii) 2

(iii) 3

(iv) When the six squares are cut from the cube, each edge of the original cube appears twice, and each vertex three times. Hence the cube has 6 faces, $24/2 = 12$ edges and $24/3 = 8$ vertices. Thus the Euler characteristic is 2.



(v) The four triangles together have 4 faces, 12 edges and 12 vertices. Thus the tetrahedron has 4 faces, $12/2 = 6$ edges and $12/3 = 4$ vertices.



For the eight triangles forming an octahedron, we find 8 faces, 24 edges and 24 vertices, but four faces share any given vertex (not three). Hence we have 8 faces, $24/2 = 12$ edges and $24/4 = 6$ vertices. In both cases, the number $F - E + V$ turns out to be 2. ■

UNIT 2
SUBDIVISIONS AND ORIENTABILITY

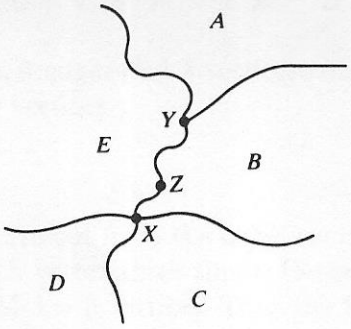
2.1 Applications of the Euler Characteristic

We have seen how to calculate the Euler characteristic of a surface and use it to prove that some surfaces are not topologically equivalent. Before continuing with our programme of classifying surfaces, it is worth spending a little time on other applications of the Euler characteristic.

The first application is to the famous map-colouring problem. Imagine a map of countries drawn on the surface of a sphere. The usual way to colour this map is so that two countries with a common frontier have different colours. Countries which meet at only a single point (or a finite number of points) can be given the same colour. The question is: how many colours are needed? It has been long conjectured that four colours would always be sufficient, and this was proved fairly recently by Appel and Haken.

We shall not be able to prove this, but, by using the Euler characteristic, it is quite easy to show that six colours are sufficient.

To relate this to what we have been doing, we need to think of a subdivision of a sphere into faces, edges and vertices in which each country forms a face, and each edge is a frontier between countries. Since the frontiers (edges) are allowed to be curved, we delete unnecessary vertices in the middle of a frontier between two countries and retain only those where three or more countries meet.



The sketch illustrates this: the vertices X and Y are necessary since three countries meet there; but Z can be omitted, so that we are left with just the edge XY and not the two edges XZ, ZY. Omitting Z does not affect the count of faces minus edges plus vertices, because it decreases both the number of edges and the number of vertices by one.

You can now read **B**'s proof. He uses terminology like "frontier", "guard" and "signpost", which may not sound very mathematical but make the proof easier to remember!

READ B from page 20, line 3 to page 22, line -4.

NOTES

(i) *Page 21, line 9*
The notation N_0, N_1, N_2. As you will have realized, **B** uses N_2, N_1, N_0 for the number of faces (F), edges (E) and vertices (V). The subscript, of course, gives the dimension — two for faces, one for edges and zero for vertices.

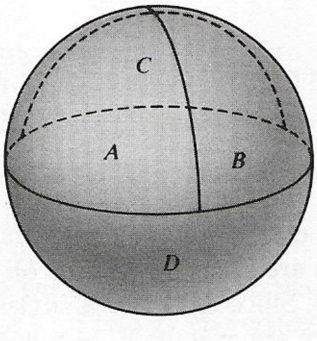
(ii) *Page 21, line -6*
"Signpost". We consider not the post but the sign pointing along each frontier. There would be four signposts at X, above, and three at Y. Each frontier thus has one signpost at each end, so there are $2N_1$ altogether.

(iii) *Page 21, last line*
The point is that $N_0 - N_1 + N_2$ is the Euler characteristic of the sphere, and so equals 2; the assumption that seven colours were needed has led to the conclusion that the Euler characteristic is ≤ 0, so the assumption must be false; that is, six colours at most will be needed. ▲

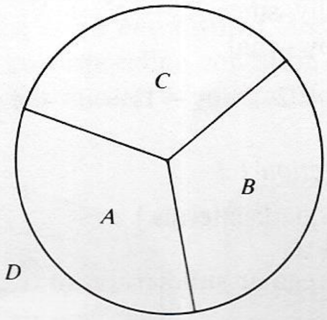
EXAMPLE 1 Find a map on the sphere which cannot be coloured with fewer than four colours.

Solution

All that is required is to realize that one has to find four regions on the sphere so that each of them has a common frontier (not just a common point) with the other three. The following diagram shows one such arrangement.



Another way to picture this is to think of the stereographic projection from the south pole. This just becomes



where D is the whole of the rest of the plane outside the circle. ☐

Another application of the Euler characteristic is to prove that only certain regular polyhedra exist. To each regular polyhedron we can associate a subdivision of the (topological) sphere into faces, edges and vertices such that each face has the same number of edges and each vertex is the end of the same number of edges. (Such a subdivision is called a *regular* subdivision.) It is quite easy to show that the condition $F - E + V = 2$ imposes conditions on the possible number of faces and edges for a regular subdivision, and hence a regular polyhedron.

Similarly, we may investigate regular subdivisions of other surfaces, although these are not related to regular polyhedra, and establish some similar results.

READ B from page 22, line -3 to page 27, line -10.

NOTE *Page 23, line -3*
A "two-sided polygon" is one with two edges (at least one of which must, of course, be curved). ▲

SAQ 1 Show that for a regular subdivision of a closed surface with Euler characteristic -2, the equation $\frac{1}{j}+\frac{1}{k}+\frac{1}{N_1}=\frac{1}{2}$ holds, where N_1 is the number of edges, j the order of each vertex and k the number of edges to each face. Deduce that if each face is triangular ($k=3$), then $j \geq 7$.

SOLUTION TO SAQ 1 $N_0 - N_1 + N_2 = -2$ and, as in the text,

$$2N_1 = kN_2, \quad 2N_1 = jN_0.$$

Hence $\left(\frac{2}{j}-1+\frac{2}{k}\right)N_1=-2$. Rearranging, we obtain $\frac{1}{j}+\frac{1}{k}+\frac{1}{N_1}=\frac{1}{2}$. If $k=3$, then $\frac{1}{j}+\frac{1}{N_1}=\frac{1}{6}$. Since $N_1 \geq 1$, $\frac{1}{N_1}$ is positive, so $\frac{1}{j}<\frac{1}{6}$.

Hence $j > 6$ and, since j is an integer, j must be at least 7. ■

EXAMPLE 2 **B** *page 49, Question 2*

Let n great circles be drawn on the sphere $x^2+y^2+z^2=1$ so that no three are concurrent. Into how many polygons do these circles divide the sphere?

Solution

Every great circle is met by each of the $n-1$ others in two points. These are the vertices. Therefore there are $2(n-1)$ vertices on each great circle (no three great circles are concurrent). Since there are n circles, and each vertex is common to two circles, there are exactly $N_0 = n(n-1)$ vertices.

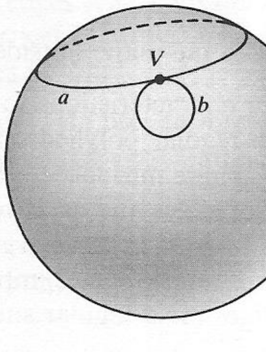
The edges of the polygons are the arcs between adjacent vertices on each great circle. $2(n-1)$ vertices divide a circle into $2(n-1)$ arcs, so there are $N_1 = 2n(n-1)$ edges. Finally, since $N_0 - N_1 + N_2 = 2$, where N_2 is the number of polygons, it follows that

$$N_2 = 2 + n(n-1) = n^2 - n + 2.$$

SAQ 2 **B** *page 49, Question 3 on Section 1.3*

(The "squares" are, in fact, quadrilaterals.)

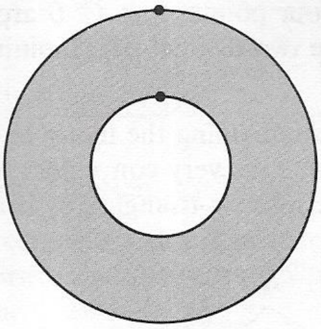
SAQ 3 Why is the following not a regular subdivision of the sphere?



SOLUTION TO SAQ 2 No. Some vertices have order 5 and some have order 4. ■

SOLUTION TO SAQ 3 Not every "face" has the same number of edges; the two circular faces have one edge, the other face has two. ■

We conclude this section on combinatorial and arithmetical results about subdivisions which can be proved using the Euler characteristic with some results for special subdivisions into triangles. These are called triangulations. To explain why triangulations are useful, we have to go back to the definition of a "face" of a subdivision. You will remember that a face has to be homeomorphic to a disc; for example, we are not allowed to call an annulus a "face".



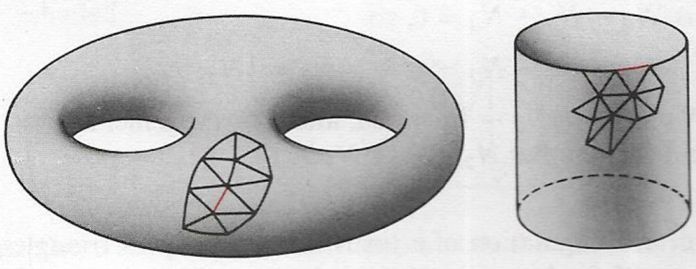
For the subdivisions we have been looking at, it is not hard to check that the faces are, indeed, homeomorphic to discs. But a complicated surface might be given to us with a complicated subdivision, and the checking might be difficult or at least tedious.

Triangulations satisfy conditions which are easy to verify and which, at the same time, guarantee that the faces are topological discs — in fact, they are (curved) triangles, as the terminology suggests.

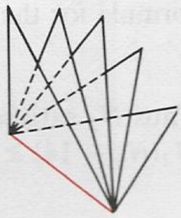
The conditions for a subdivision to be a *triangulation* are as follows.

1. Every face has exactly 3 edges and 3 vertices.
2. Every edge has exactly 2 vertices.
3. There is at most one face with three given vertices.
4. There is at most one edge with two given vertices.

Condition 1 is the main condition which ensures that each face is homeomorphic to a disc. Condition 2 is the corresponding condition for edges. The third condition is a technical convenience; it means that we can "label" a face by specifying its vertices, as no two distinct faces can have the same vertices. Condition 4 is the corresponding condition for edges. One consequence of our definition of a surface is that an edge of a triangulation is an edge of at most two faces.

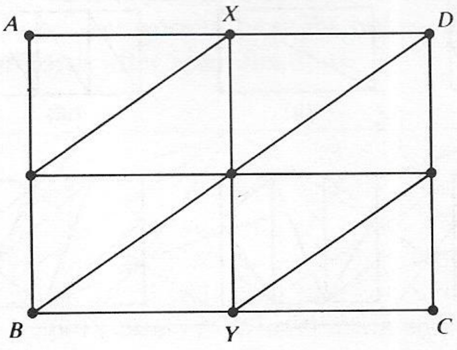


Allowed



Not allowed

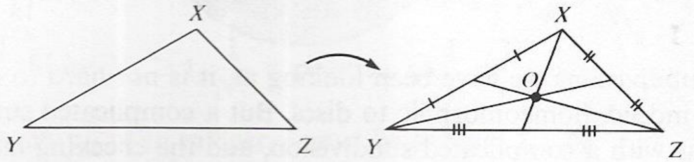
EXAMPLE 3 Check that the following diagram is a triangulation of the rectangle *ABCD* (a topological disc). If the edges *AB*, *DC* and *AD*, *BC* are identified to make a torus, why is it not a triangulation of the torus? Subdivide the triangulation of *ABCD* further, so that when the opposite edges are identified, the resulting subdivision of the torus *is* a triangulation.



Solution

It is easy to check conditions 1–4 for the diagram. However, when opposite edges are identified, the four points A, B, C, D are identified to a single point and hence there would be two distinct edges joining X to this point in the torus, violating condition 4.

There are many ways of subdividing the figure to obtain a triangulation of the torus (after identifications). One very convenient way of doing it is to subdivide each triangle in the figure into six triangles, by drawing the medians.



This is called barycentric subdivision. It may seem an extravagant way to subdivide, but it is easy to describe!

EXAMPLE 4 Show that every triangulation of the torus must have at least 14 triangles.

Solution

Suppose we have a triangulation with N_0 vertices, N_1 edges and N_2 triangles. Chop up the torus into the N_2 triangles. Each triangle has 3 edges, so we have $3N_2$ edges altogether in the separate pieces. But each edge was common to two triangles before the torus was chopped up, so $2N_1 = 3N_2$.

Since there is at most one edge joining a pair of vertices, N_1 cannot be more than the number of pairs of distinct vertices ($^{N_0}C_2$), i.e.

$$N_1 \leq \tfrac{1}{2}N_0(N_0 - 1).$$

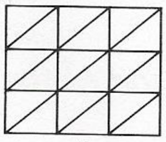
Using $2N_1 = 3N_2$ we can write this as $3N_2 \leq N_0(N_0 - 1)$. Now, from the Euler formula for the torus, $N_0 - N_1 + N_2 = 0$, so

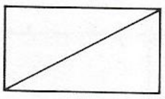
$$N_0 = N_1 - N_2 = \tfrac{3}{2}N_2 - N_2 = \tfrac{1}{2}N_2.$$

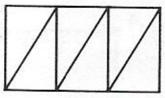
Thus N_2 satisfies $3N_2 \leq \frac{1}{2}N_2(\frac{1}{2}N_2 - 1)$, which, after rearrangement, gives $N_2(N_2 - 14) \geq 0$, and this implies $N_2 \leq 0$ or $N_2 \geq 14$. ☐

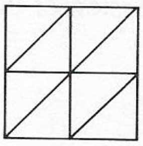
There is a diagram of a triangulation of a torus using exactly 14 triangles on **B** page 50. (See **B** page 25 for how to make this hexagon into a torus.) Such a triangulation is called *minimal*, for obvious reasons.

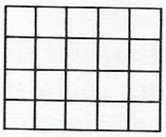
SAQ 4 Which of the following are triangulations of a torus after the opposite edges of the outer rectangles have been identified?

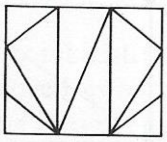

(i)

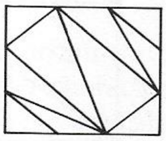

(ii)


(iii)

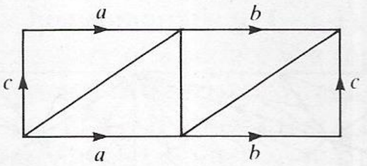

(iv)


(v)


(vi)


(vii)

SAQ 5 The following diagram represents a subdivision of a torus.



Subdivide it so that it is a triangulation.

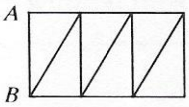
SAQ 6 Considering the torus as a rectangle with opposite edges identified, find a regular subdivision of the torus into N vertices, $3N$ edges and $2N$ triangles which is also a triangulation.

SOLUTION TO SAQ 4

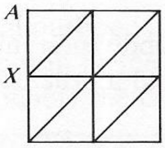
(i) is a triangulation. The others are not.

(ii) is not, because the vertices of each triangle are all the same point after the identifications are made.

(iii) is not, because after the identifications, the edge AB has only one vertex.

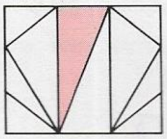


(iv) is not, because after the identifications, there are two distinct edges joining A and X.

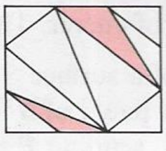


(v) is not a subdivision into triangles!

(vi) Some of the triangles do not have three distinct vertices (e.g. the one shaded).

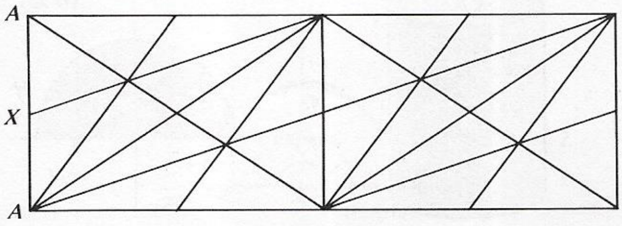


(vii) There are two distinct triangles with the same vertices (shaded).

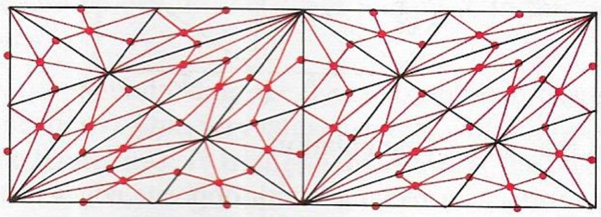


■

SOLUTION TO SAQ 5 Subdividing each of the two triangles into six smaller ones, as in Example 3, is not quite enough. We hope you were not caught out — there are two edges joining A and X on the torus after identifications.



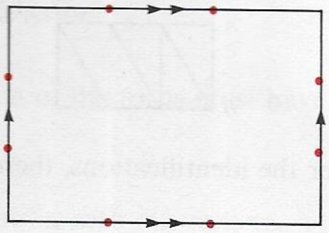
However, if we apply barycentric subdivision again, this time to the figure above, all is well. Of course, there are many other possible ways, but this is the easiest to describe.



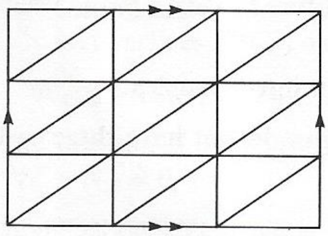
■

SOLUTION TO SAQ 6 We have seen that regular subdivisions, described on **B** page 27, with $N = 1, 2, 3, 4$ are not triangulations. ($N = 1, 3, 4$ are, respectively, parts (ii), (iii) and (iv) of SAQ 4. $N = 2$ is illustrated in SAQ 5.)

The regular subdivision must have at least two vertices on each side of the rectangle, in addition to the corners. This will avoid having two edges joining the same two vertices after identifications.



Hence we can expect that the subdivision into 18 triangles ($N = 9$) will be a triangulation in the full sense, and a little checking verifies that this is true.

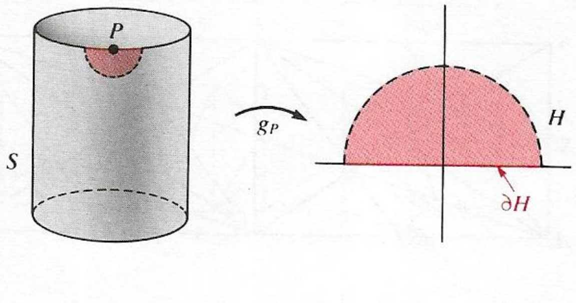


■

2.2 Surfaces with Boundary

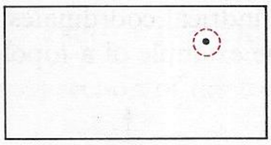
We saw in *Unit 1* that the number of boundary curves of a surface S is a topological invariant; we did not justify the assertion that the boundary does, in fact, consist of a finite number of curves. This justification is our next task.

Recall that a boundary point of a surface S is a point P where we can find a homeomorphism g_P mapping a patch of S to the half disc $H = \{(x, y): x^2 + y^2 < 1, y \geq 0\}$, sending P to the origin $(0, 0)$. (By a "patch" we mean an open subset of S containing P.) It is not possible to find such a homeomorphism with the whole disc $D = \{(x, y): x^2 + y^2 < 1\}$.



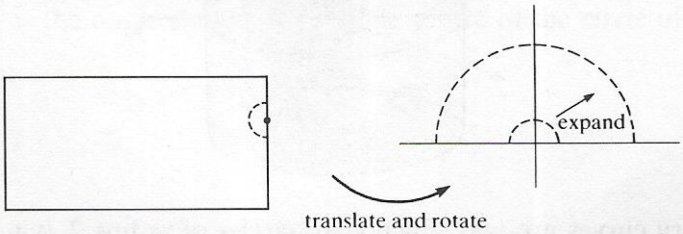
Thus, near a boundary point P, the surface S "looks" (to the topologist) exactly like the half disc H. Now, the boundary of H is the set of points $\{(x,0)\colon -1 < x < 1\}$, which is homeomorphic to $(-1,1)$. It is true (though we shall not prove it) that the boundary of a surface is a topological invariant. That is, a homeomorphism between two surfaces S_1 and S_2 maps the boundary ∂S_1 of S_1 into the boundary ∂S_2 of S_2. It follows that g_P maps a portion of ∂S near P homeomorphically onto ∂H and sends P to $(0,0)$. Hence the boundary of S near P looks exactly like the boundary of H near $(0,0)$. In other words, the set of points of ∂S near P is homeomorphic to $(-1,1)$, and the homeomorphism maps P to $(0,0)$. This means that the boundary ∂S is a collection of curves. None of these curves has an endpoint, because at such a point we would be unable to find a surrounding portion of boundary homeomorphic to $(-1,1)$. There are no self-intersections either, because a portion of a curve near a self-intersection is not homeomorphic to an interval. The only curves with no endpoints and no self-intersections are homeomorphic to circles, and so ∂S consists of a number of curves, each homeomorphic to a circle. The number has to be finite, because S is compact.

EXAMPLE 5 Verify that a rectangle is a surface with boundary,

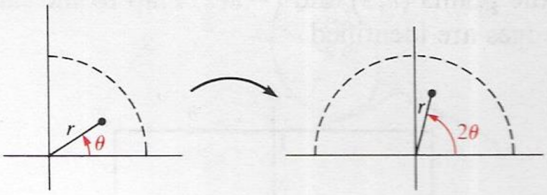


Solution

Any interior point clearly has neighbourhoods homeomorphic to D. It is easy to see how to construct a homeomorphism to the half disc H from any point of the edge of the rectangle, except possibly the corners.



For the corners, we use the same idea, but we must then show how to map the quarter disc onto the half disc.

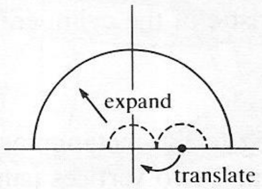


This can be done by mapping the point with polar coordinates (r,θ) where θ lies between 0 and $\pi/2$, to the point $(r, 2\theta)$.

The boundary curve is rectangular, but is homeomorphic to a circle (cf. Example 5 of *Unit 1*).

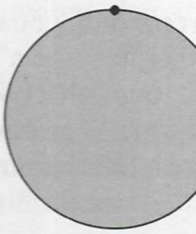
SAQ 7 Give an explicit homeomorphism from a patch of H to the half disc H carrying the point $(\frac{1}{2},0)$ to $(0,0)$.

SOLUTION TO SAQ 7

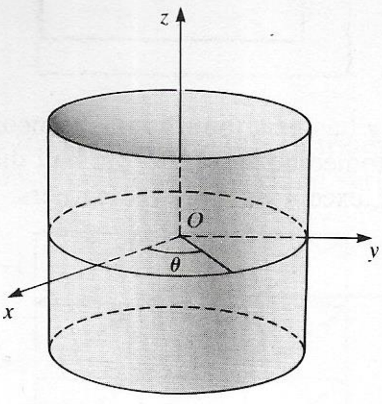


A half disc of radius $\frac{1}{4}$, centre $(\frac{1}{2}, 0)$, is a suitable patch. First translate to the origin (by the homeomorphism $(x,y) \longmapsto (x-\frac{1}{2}, y)$). Then "scale up" to fill the whole of H (by the homeomorphism $(x,y) \longmapsto (4x, 4y)$). The composition of these, $(x,y) \longmapsto (4x-2, 4y)$, is a homeomorphism and fulfils our requirements. ■

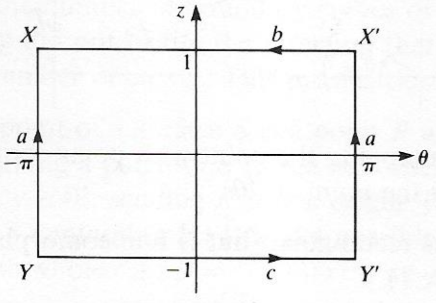
The simplest example of a surface with boundary is, perhaps, the disc $\{(x,y): (x^2+y^2) \leq 1\}$ with boundary the unit circle. By subdividing it with a vertex on the boundary, we obtain a subdivision with one face, one edge and one vertex, so that the Euler characteristic is 1.



Another example of a surface with boundary is the cylinder. If we consider the set of points in $\mathbb{R}^3$ whose cylindrical coordinates (r, θ, z) satisfy $r = 2$ and $-1 \leq z \leq 1$, this gives us one example of a topological cylinder.



The boundary curves are two (Euclidean) circles of radius 2, lying respectively in the planes $z = 1$ and $z = -1$. If we plot the coordinates θ and z as rectangular coordinates in the plane, with $-\pi \leq \theta \leq \pi$, we obtain a rectangular map. For each z, the points (π, z) and $(-\pi, z)$ map to the same point, so the opposite vertical edges are identified.



Thus we can represent the cylinder as a quadrilateral with edge equation $aba^{-1}c = 1$.

EXAMPLE 6 Compute the Euler characteristic of the cylinder.

Solution

The subdivision corresponding to the rectangular map has 1 face, 3 edges (since XY and $X'Y'$ are identified) and two vertices (since X, X' and Y, Y' are identified).

Thus the Euler characteristic is $1 - 3 + 2 = 0$, the same as for the torus.

Note that we *cannot* conclude that the cylinder and the torus are homeomorphic. Remember that the classification theorem requires three conditions for homeomorphism. One condition is that the number of boundary curves be equal — 0 for the torus, 2 for the cylinder. ☐

We now turn to another surface, with just one boundary curve, which is of fundamental importance – the Möbius band. We shall see in the next section that the Möbius band is central to a discussion of orientability — the third of the three topological invariants used in classifying surfaces. The reading passage starts with the cylinder we have just described, and places it inside the torus we met in *Unit 1*.

READ B page 29, last paragraph to page 33, line 10 and Section 1.5, page 35 as far as page 36, line 6.

NOTES (i) *Page 30, line 3*

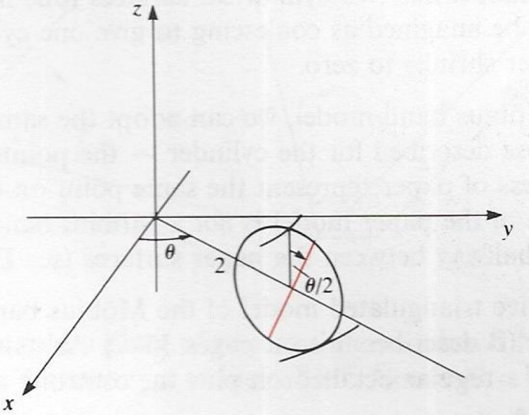
Notice that the equation

$$z \sin\frac{\theta}{2} = (r - 2)\cos\frac{\theta}{2},$$

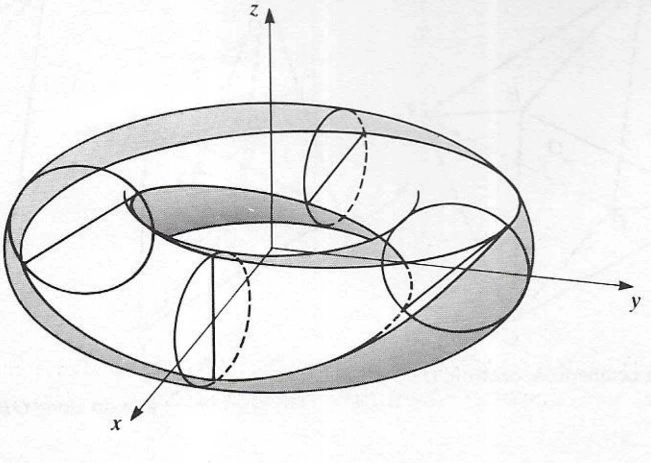
when restricted to the cross-section of the torus at longitude θ, represents a diameter with gradient

$$\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

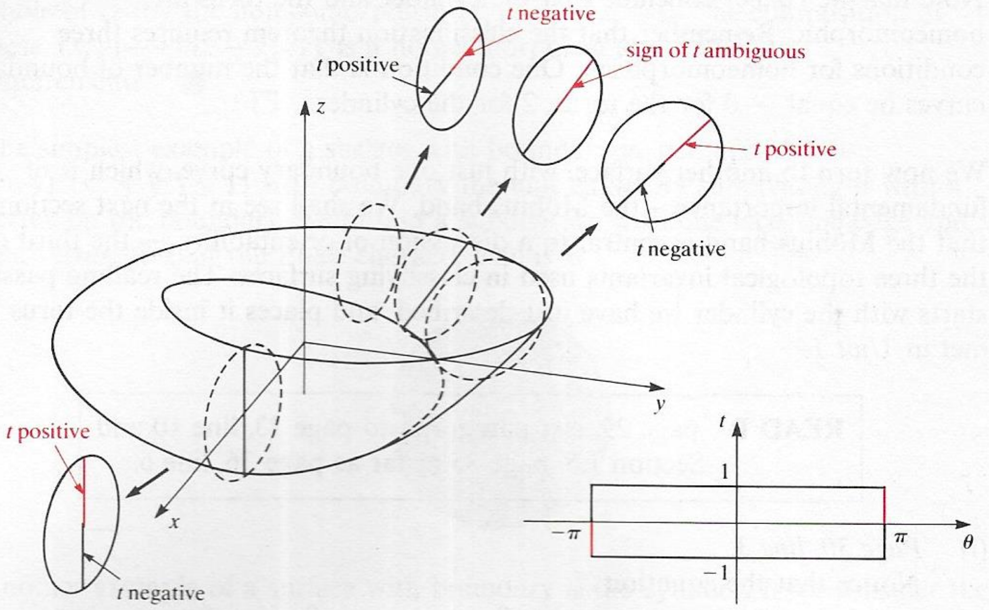
Hence $\frac{\theta}{2}$ is the angle shown below. The centre of the circle of cross-section is $(2, \theta, 0)$.



Thus the diameter is vertical at $\theta = 0$, horizontal at $\theta = \pi$, and makes a *half* turn as θ increases from 0 to 2π.



(ii) *Page 30, last paragraph*
You might find the following diagram helpful in understanding the role of t.

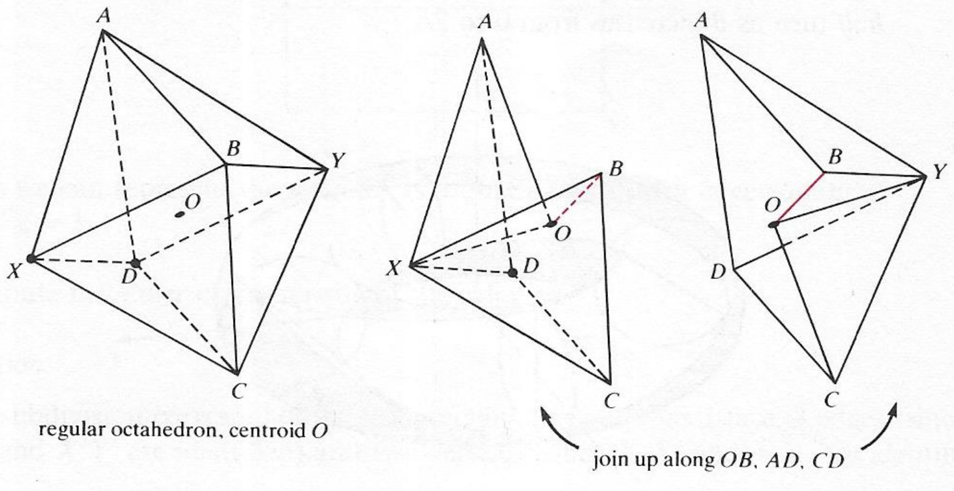


(iii) *Page 32, line 18*
"Closed surface". This use of the word closed, while standard, must not be confused with the meaning of closed in "closed set" defined on page 18 of *Unit 1*.

(iv) *Page 32, line −12*
(The paper models.) When we make a paper model of a cylinder, we use the convention that two points on opposite sides of the paper thickness actually represent a *single* point on the cylinder — that is, the cylinder is the "idealized" version of the paper model obtained when the thickness of the paper shrinks to zero. We could express this differently by saying that the paper model has two cylindrical surfaces (one inside and one outside) which can be imagined as coalescing to give one cylinder as the thickness of the paper shrinks to zero.

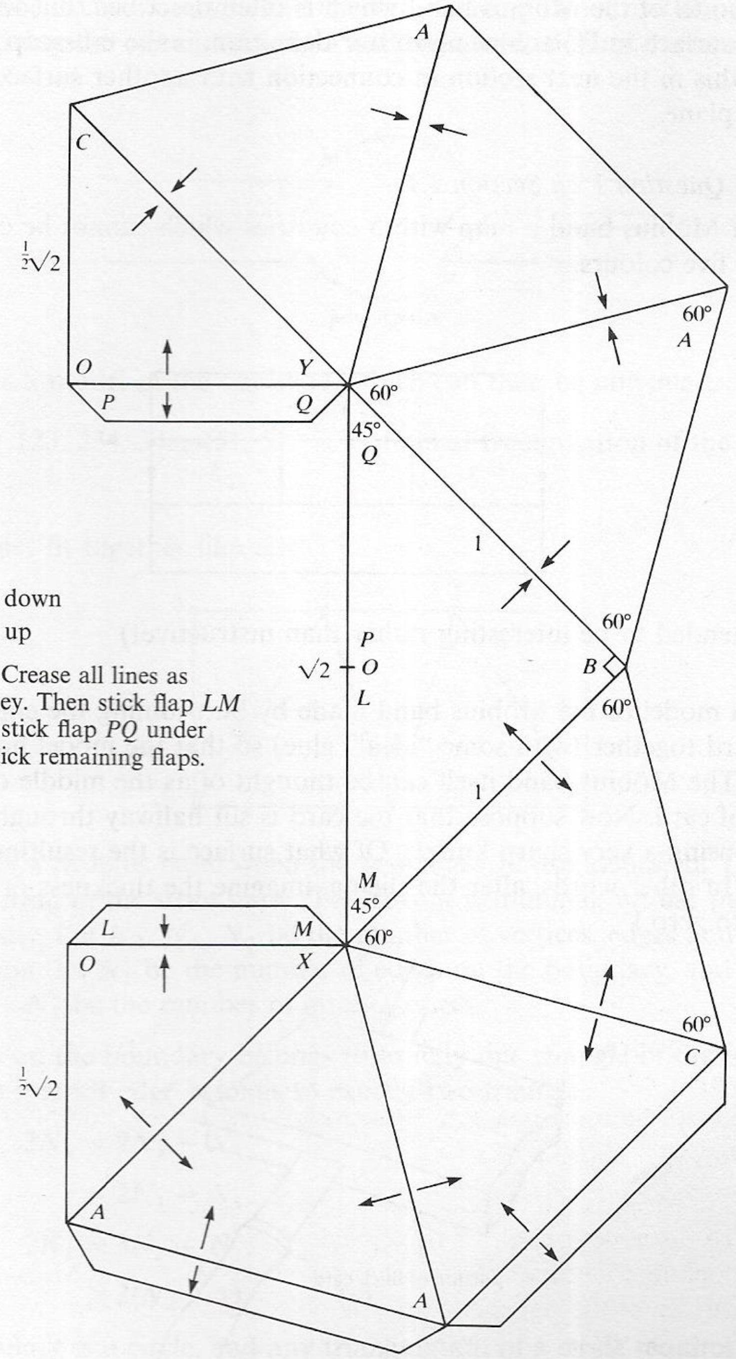
For the Möbius band model, we can adopt the same convention as the first of those described for the cylinder — the points on opposite sides of the thickness of paper represent the same point on the Möbius band. But the surface of the paper model is *not* a Möbius band: the Möbius band really lies halfway between the paper surfaces (see Example 8). ▲

A rather attractive triangulated model of the Möbius band can be constructed from 9 triangles (**B** describes this on pages 36–7). As its vertices, the model has the 6 vertices of a regular octahedron plus the centroid, and one way to think of it is shown below.



regular octahedron, centroid O

join up along OB, AD, CD


A
C
$\frac{1}{2}\sqrt{2}$
O
Y
P
Q
60°
45°
Q
60°
A
1
60°
P
$\sqrt{2}$
O
B
L
60°
1
M
45°
L
M
60°
O
X
60°
C
$\frac{1}{2}\sqrt{2}$
A
A

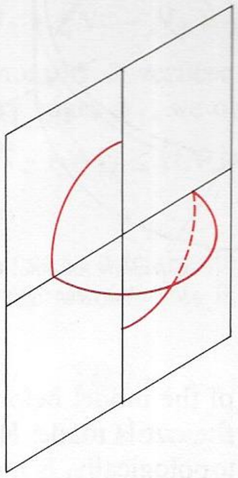
Key

←|→ bend down

→|← bend up

Instructions Crease all lines as indicated in key. Then stick flap LM over XO and stick flap PQ under OY. Finally stick remaining flaps.

To make one, cut and fold a piece of paper as shown above. Notice that the boundary curve is the perimeter of triangle ABC. Notice also the shape of the model near O.

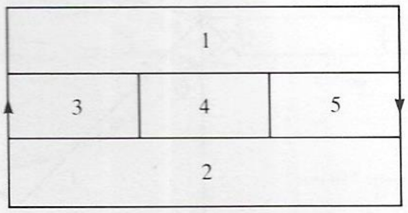


Another model of the Möbius band which is often described, but which is not actually a surface in $\mathbb{R}^3$ according to our definition, is the crosscap. We shall deal with this in the next section in connection with another surface, the projective plane.

EXAMPLE 7 **B** *page 51, Question 1 on Section 1.4*

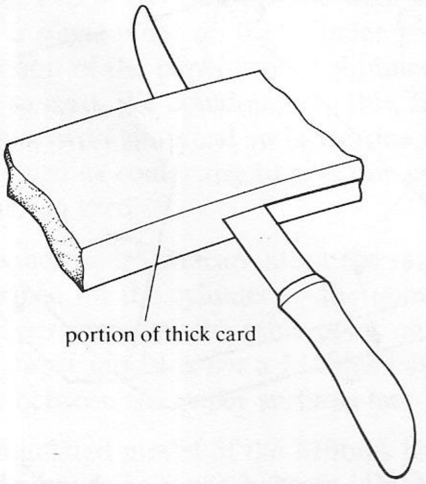
Draw on a Möbius band a map with 5 countries which cannot be coloured with fewer than five colours.

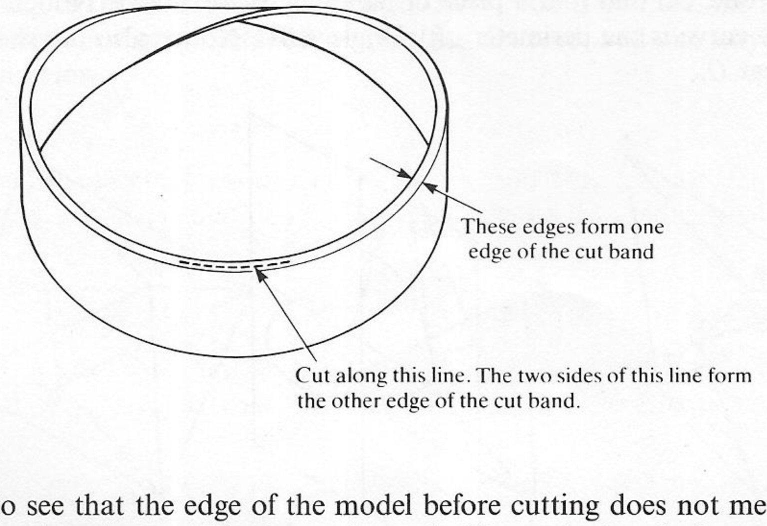
Solution



(This is intended to be interesting rather than instructive!)

EXAMPLE 8 Consider a model of the Möbius band made by butt-joining the ends of a strip of thick card together (with some “ideal” glue) so that the model has uniform thickness. The Möbius band itself can be thought of as the middle of the thickness of card. Now suppose that the card is slit halfway through its thickness (using a very sharp knife!). Of what surface is the resulting card shape a model? (In other words, after the slicing, imagine the thickness of the card as shrinking to zero.)

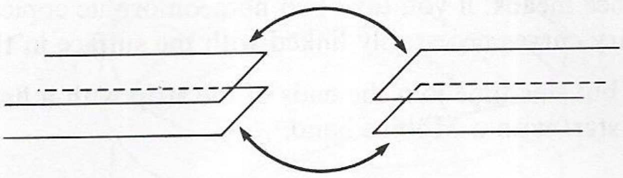

portion of thick card


These edges form one edge of the cut band
Cut along this line. The two sides of this line form the other edge of the cut band.

Solution

It is easy to see that the edge of the model before cutting does not meet the two sides of the edge along which the cut is made. Hence the band after cutting has two boundary curves — and, topologically, is a cylinder. (It has *four* half twists.

If you do not believe this, you can check by placing two strips of paper together, giving the combined strip a half twist and sticking the two pieces together.)

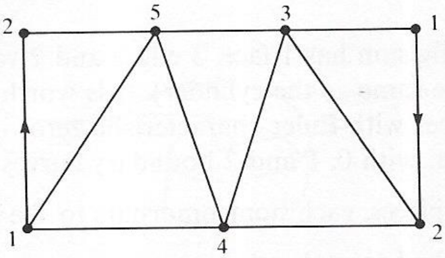


This makes a model of the cut band, which can then be unwound.

EXAMPLE 9 Prove that 123, 234, 345, 451, 512 is a minimal triangulation of the Möbius band.

Solution

The triangles fit together like this:



which gives a Möbius band when the two edges 12 are identified. This is clearly a triangulation in the strict sense. To prove it is minimal, we use the Euler characteristic. Let N_0, N_1, N_2 be the number of vertices, edges and faces in any triangulation; let N_1' be the number of edges on the boundary, and let $N_1'' = N_1 - N_1'$ be the number of interior edges.

Each edge on the boundary belongs to exactly one triangle of the triangulation, while each interior edge belongs to exactly two triangles.

$$\begin{aligned} \text{Therefore}\quad 3N_2 &= 2N_1'' + N_1' \\ &= 2N_1 - N_1'. \\ \text{So}\quad 2N_1 &= 3N_2 + N_1'. \\ &\geq 3(N_2 + 1). \qquad (*) \end{aligned}$$

(The boundary is a circle, and any triangulation of a circle requires at least three vertices — e.g. only two vertices would lead to two distinct edges joining the same two vertices. Hence $N_1' \geq 3$.)

Also, $N_0 - N_1 + N_2 = 0$, since the Euler characteristic is 0, so that $N_0 = N_1 - N_2$. Using $2N_1 = 3N_2 + N_1'$, we find

$$N_0 = \tfrac{3}{2}N_2 + \tfrac{1}{2}N_1' - N_2 = \tfrac{1}{2}(N_2 + N_1').$$

Since N_0 is not less than the number of vertices on the boundary, which also equals the number of boundary edges N_1', we obtain $N_1' \leq N_0$. Therefore

$$N_0 = \tfrac{1}{2}(N_2 + N_1') \leq \tfrac{1}{2}(N_2 + N_0)$$

$$\text{or}\qquad N_0 \leq N_2. \qquad (**)$$

Further, $N_1 \leq \frac{1}{2}N_0(N_0 - 1)$, since in a triangulation there cannot be more edges than distinct pairs of vertices. Combining this inequality with $(*)$ and $(**)$ we have

$$\tfrac{3}{2}(N_2 + 1) \leq N_1 \leq \tfrac{1}{2}N_0(N_0 - 1) \leq \tfrac{1}{2}N_2(N_2 - 1).$$

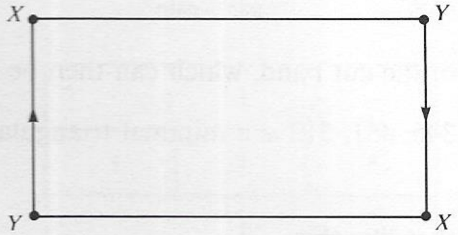
Rearranging gives $N_2^2 - 4N_2 - 3 \geq 0$. Hence, since N_2 is a positive integer, N_2 must be at least 5.

SAQ 8 Compute the Euler characteristic of the Möbius band from the flat representation as a rectangle.

SAQ 9 **B** *page 51, Question 1 on Section 1.5*

(The last sentence means: if you take two homeomorphic copies of a surface, are the boundary curves necessarily linked with the surface in the same way?)

SAQ 10 Repeat SAQ 9, but this time join the ends of the strip with a half twist (by π radians) — i.e. start with a Möbius band.



SOLUTION TO SAQ 8 The rectangular subdivision has 1 face, 3 edges and 2 vertices so the Euler characteristic is 0 (the same as the cylinder). It is worth noting that we have now seen three surfaces with Euler characteristic zero — the torus, the Möbius band and the cylinder, with 0, 1 and 2 boundary curves respectively. ■

SOLUTION TO SAQ 9

(a) There are three pieces, each homeomorphic to the original.

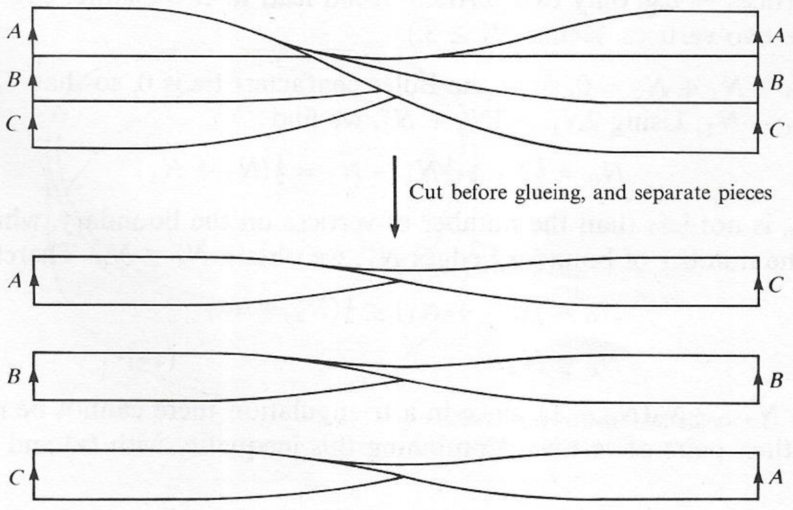
(b) Each piece is linked to each other piece.

(c) Two.

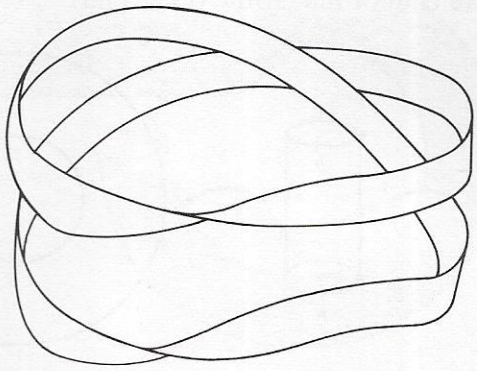
(d) A cylinder.

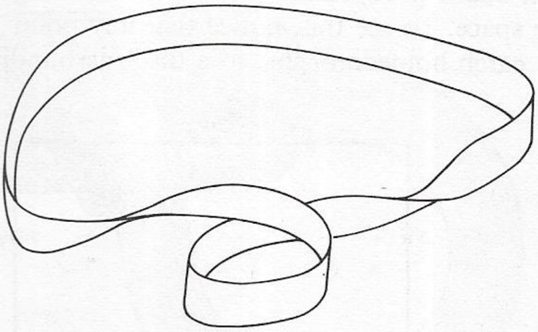
(e) No — for example, the boundary circles of an ordinary cylinder are not linked, whereas those of a cylinder formed by joining the ends of a strip with a full twist (2π radians) are. ■

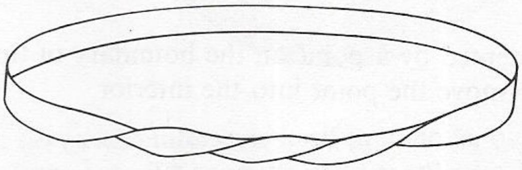
SOLUTION TO SAQ 10 The piece that was originally the centre of the band must be a Möbius band itself, because if the strip had been cut before being glued, and the outer pieces removed, we would be left simply with a strip of one-third the width which would be glued after being given a half twist. The other piece must be made up of two strips, each with a half twist, glued together as shown below.



Now observe that we can match the edges C and then glue the edges A with no further twisting. Thus the outer piece is a cylinder, although the method of construction causes it to have four half twists.



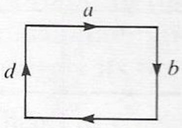




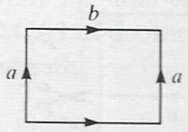
■

2.3 Surfaces that are Not Subsets of $\mathbb{R}^3$

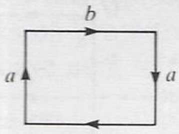
So far we have looked at surfaces which are subsets of ordinary three-dimensional Euclidean space, $\mathbb{R}^3$. We have seen that they can be represented by polygons with identified edges (the identifications being indicated conveniently by letters and arrows). This method of representation has given us five surfaces which can be represented by a square, or rectangle, with identifications.



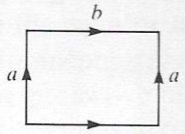
disc



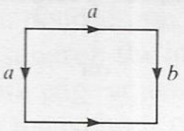
cylinder



Möbius band

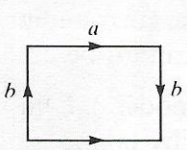


torus

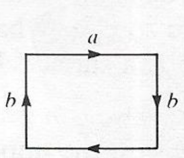


sphere

There are two further possibilities.

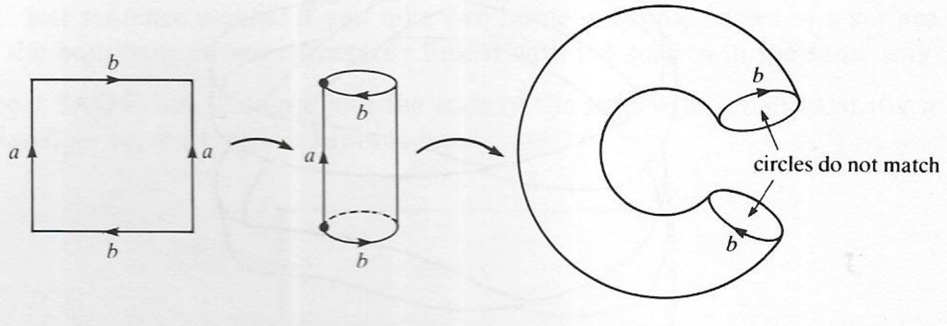


(i)

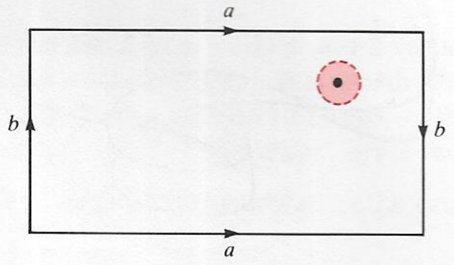


(ii)

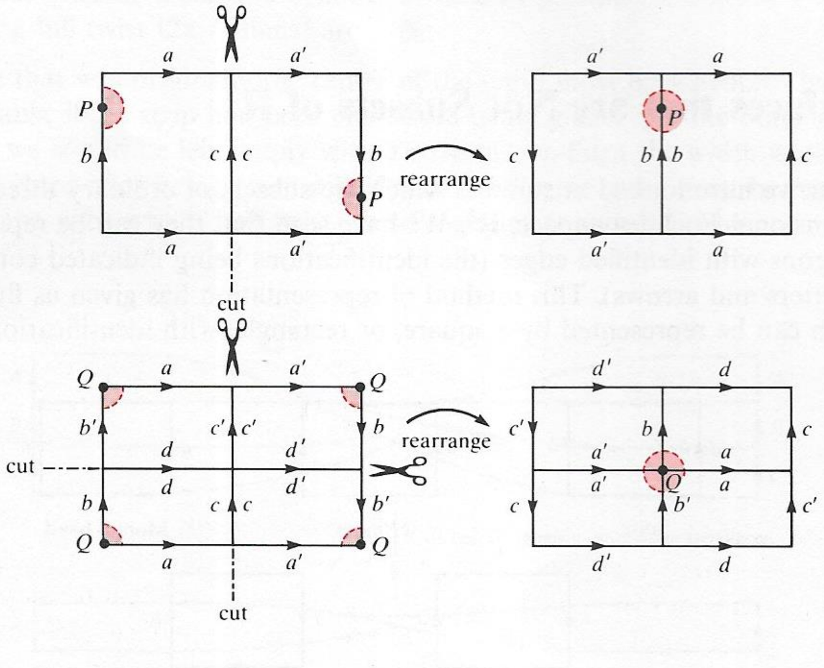
Suppose we look at (i) above and try to use this rectangle as a recipe for constructing a surface. We get stuck, as a little experiment shows. If we match one pair of edges, we cannot match the other pair.


b
a
a
b
a
b
b
b
b
circles do not match

Yet the diagram ought to represent a surface, according to our definition, as a locally disc-like space. To see this, notice that any point in the *interior* of the rectangle has a patch homeomorphic to a disc surrounding it.

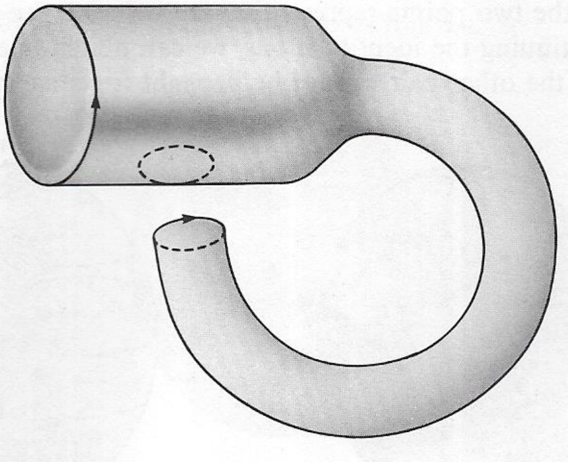


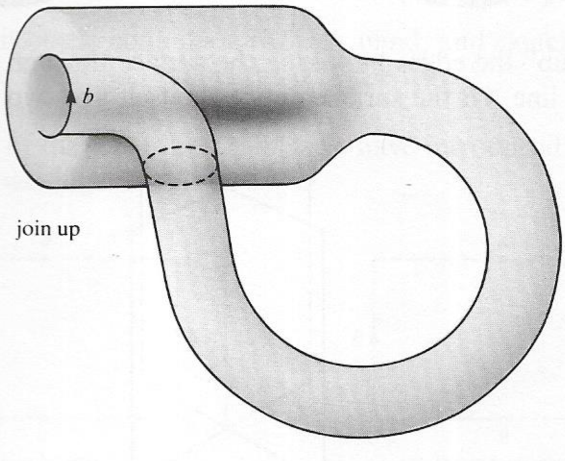
For a point represented by a point on the boundary of the rectangle, we simply cut and re-glue to move the point into the interior.


a
a'
P
b
c
c
b
P
a
a'
cut
rearrange
a'
a
c
P
b
b
c
a'
a
Q
a
a'
Q
b'
c'
c'
b
d
d'
cut
d
d'
b
c
c
b'
Q
Q
a
a'
cut
rearrange
d'
d
c'
b
c
a'
a
a'
Q
a
c
b'
c'
d'
d

Cutting the rectangle and reassembling it appropriately shows that (i) can also be represented by a different rectangle (with identifications) in which the point is in the interior, and so clearly has a disc surrounding it in the surface. So (i) is locally disc-like, and *is* a surface. But it cannot be made in $\mathbb{R}^3$.

Notice that the nearest we can get to a model of (i) in $\mathbb{R}^3$ is by trying to match the lower little circle with the upper one by taking it "round the back" of the upper one.



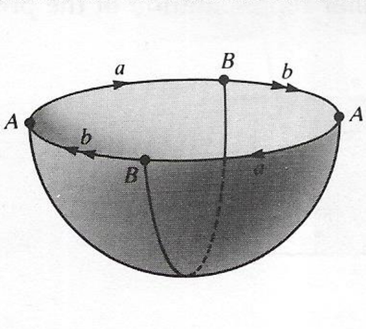


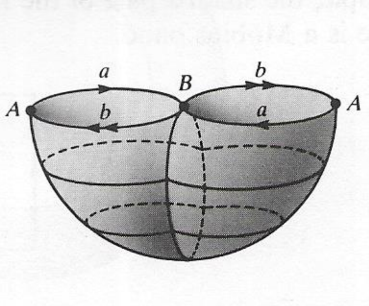
This is not a surface, because it intersects itself in a circle, but it does give some insight into what (i) represents. The surface of (i) is called a Klein bottle, after its inventor.

It is, in fact, possible to construct a surface, as a set of points in $\mathbb{R}^4$, which can be cut up and flattened into a polygon equivalent to (i); however, for everything we want to do in this course, the representation (i) is sufficient. We shall, therefore, work with representations like (i) and (ii), relying on the fact that these do satisfy our definition of a surface. Even though they do not correspond to a surface in $\mathbb{R}^3$, they do exist as higher-dimensional loci (cf. **B** page 52, Question 1).

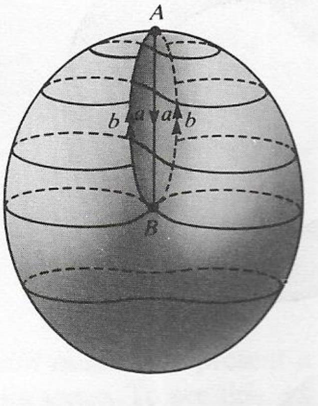
We now turn to the surface given by (ii). It is called the *projective plane* and is extremely important. In the next unit we shall show how this representation is related to the definition given in *M203* and other books on projective geometry. We shall investigate its properties by the methods we have used so far.

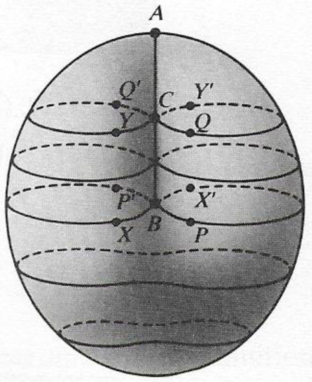
It is more difficult to construct a "fudged" model of the projective plane in $\mathbb{R}^3$ than it was for the Klein bottle. One attempt starts with the lower hemisphere of the sphere, on which the equator has been divided into four arcs. This can be flattened out to a disc (homeomorphic to a square); the identifications in the representation (ii) of the projective plane, with edge equation $abab = 1$, correspond to the identifications marked on the equator of the hemisphere.



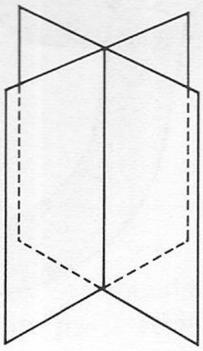


If we identify the two points representing the vertex B, we obtain the figure on the right. Continuing the identifications, we can match one pair of edges, such as a, but then the other pair cannot be brought together in $\mathbb{R}^3$.

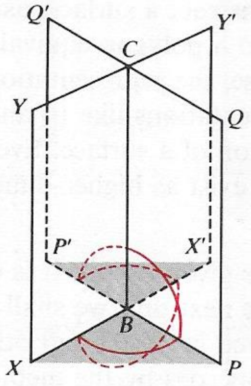




We can "join up" the edges by taking the surface through itself along AB, so that along the line AB the surface intersects itself like two planes.

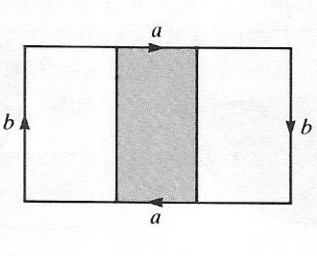


Exactly what happens at the points A and B is difficult to visualize. Near B, for instance, the model is homeomorphic to the figure below.

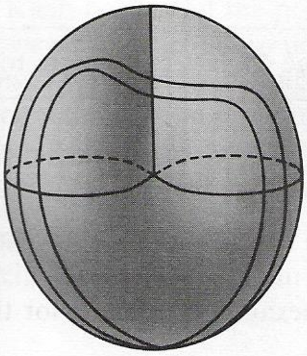


However, in the projective plane itself, the points corresponding to the planes $PQQ'P'$ and $XYY'X'$ have only B in common. Topologically, the figure is a disc surrounding B. Near A, the picture is the same.

The top half of this model — known as the *crosscap* — may be difficult to visualize properly, but it is, nevertheless, useful for a number of purposes. For example, the shaded part of the rectangular representation of the projective plane is a Möbius band.



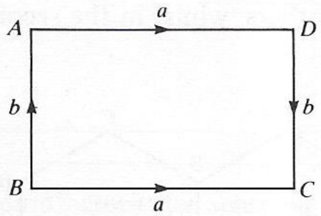
We can draw a Möbius band in the crosscap as follows.



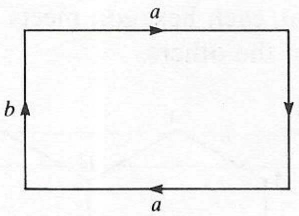
SAQ 11 Compute the Euler characteristic of the Klein bottle and the projective plane.

SAQ 12 Show that the Klein bottle contains a Möbius band, and sketch a Möbius band on the model of the Klein bottle illustrated in the text.

SOLUTION TO SAQ 11 The representation of the Klein bottle that we have introduced is a subdivision with one face and two edges.



Klein bottle

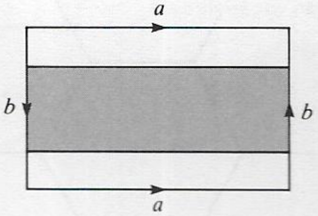


Projective plane

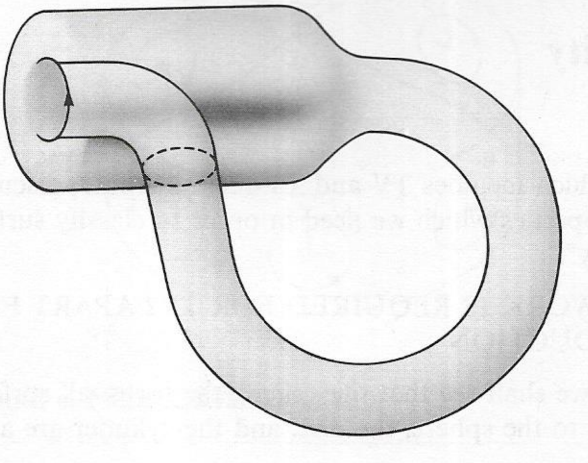
It has one vertex, because when the edges a are identified, A is identified with B, and C with D. Identifying the edges b brings all four points together. Hence the Euler characteristic is 0.

For the projective plane, we have one face and two edges, but this time two vertices. Therefore the Euler characteristic is 1. ■

SOLUTION TO SAQ 12



The shaded area — half the Klein bottle — is a Möbius band when the edges are identified. Therefore half the model of the Klein bottle is a Möbius band, though it has a semicircle where it intersects itself.



■

We conclude this section with a brief discussion about colourings and triangulations of the Klein bottle and the projective plane.

READ B from page 45, last paragraph, to the end of Section 1.6.

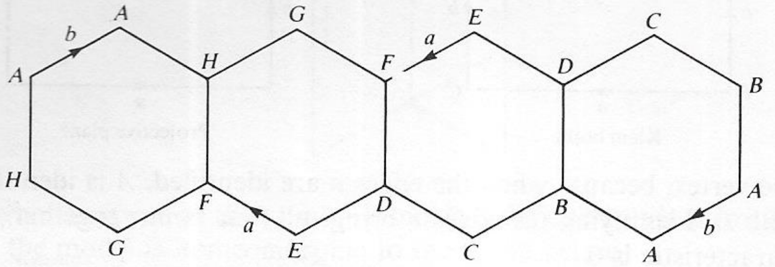
SAQ 13 Prove that there are maps on a Klein bottle that require five colours, using what we know about the Möbius band.

SAQ 14 Show that the Klein bottle has at least two "essentially different" regular subdivisions into the same number of hexagons (i.e. the number of hexagonal "neighbours" of a typical hexagon is different for the two subdivisions).

SOLUTION TO SAQ 13 We have seen that there are maps on the Möbius band requiring 5 colours. We also know that the Klein bottle contains a Möbius band, so we can construct a map on the Klein bottle by first embedding the Möbius band (with a map on it) and then regarding the complement of the band as another "country". The resulting map needs at least as many colours as the original one on the band (i.e. 5). ■

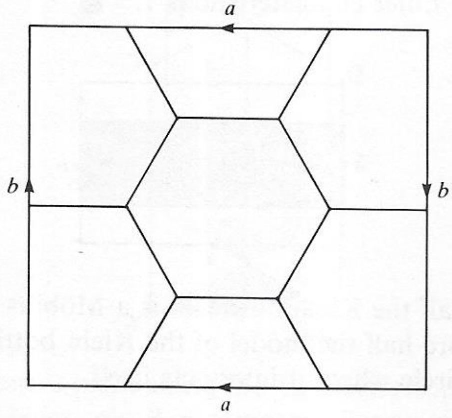
SOLUTION TO SAQ 14 There are several possible solutions. A simple one is given below — in the first subdivision, each hexagon meets two of the others, whilst in the second it meets all three of the others.

(i)



(The edge *a* has vertices *ABCDEFGH*.)

(ii)



2.4 Orientability

Introduction

This section, which includes TV and TV-related Audio, discusses the third of the topological properties which we need in order to classify surfaces: the property of "orientability".

NO PRE-TV WORK IS REQUIRED FOR TV2 APART FROM READING THIS INTRODUCTION

In this section we shall see that the sphere, the torus, all surfaces obtained by adding handles to the sphere, the disc, and the cylinder are all orientable. The

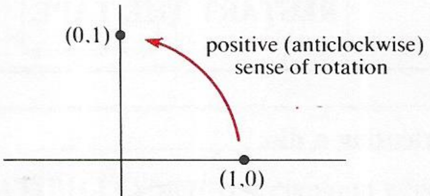
Möbius band, the Klein bottle and the projective plane are not orientable. **B** defines a non-orientable surface to be one which contains a Möbius band, and we have seen that both the Klein bottle and the projective plane do, in fact, contain a Möbius band. However, this definition does not give much insight into what orientability *means.* TV2 provides this insight with a more fundamental definition of orientability. The programme ends by demonstrating that a non-orientable surface without boundary cannot be embedded in $\mathbb{R}^3$.

After watching the TV programme, look at Frame 1 and start the first TV-related Audio Cassette at Section 2.

TV Overview

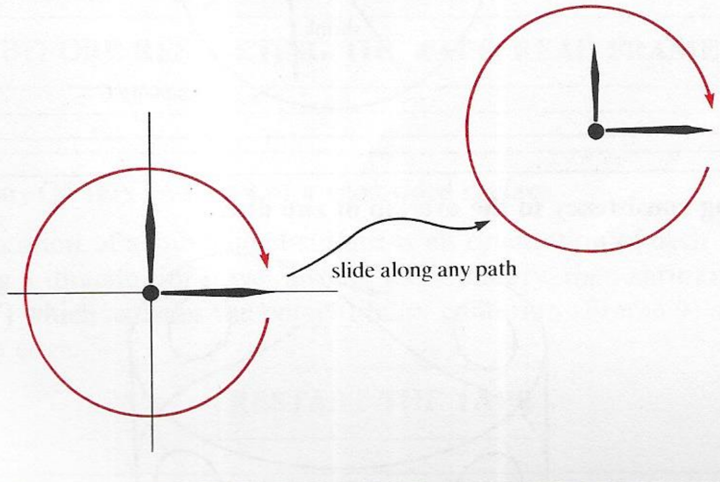
FRAME 1

Sense of rotation in $\mathbb{R}^2$



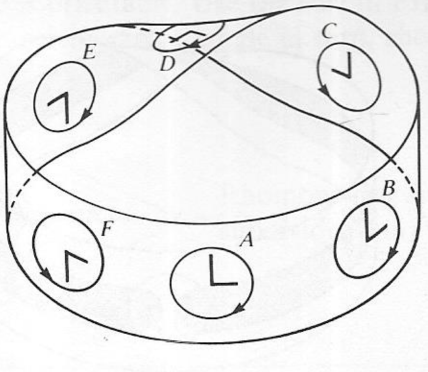
FRAME 2

"Standard" clockwise sense on the plane



FRAME 3

Sense of rotation can REVERSE on the Möbius band



The Möbius band is NON-ORIENTABLE.

FRAME 4 Definition of orientability (cf. B's definition on page 35)

(i) *An orientation of a surface*

If the process of sliding clockfaces on a surface assigns a consistent sense of rotation about each point, then that sense of rotation — over the whole surface — is called an orientation of the surface.

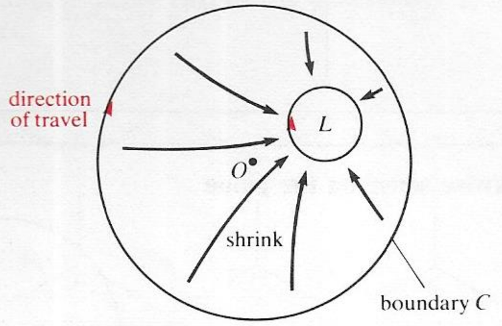
Any process or rule which assigns such a consistent sense of rotation defines an "orientation" of the surface — i.e. the process does not have to be "sliding clockfaces".

(ii) *Orientable and non-orientable*

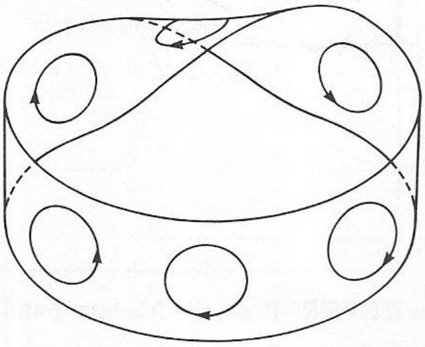
A surface is called orientable if it can be given an orientation, and non-orientable if it cannot be given an orientation.

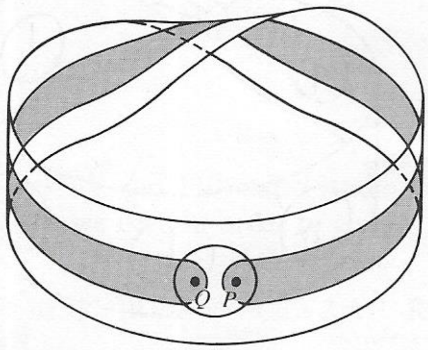
RESTART THE TAPE

FRAME 5 Different rule for orienting a disc



FRAME 6 Checking consistency in the overlap of two discs

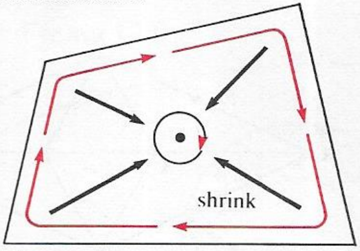




If the orientations agree at P, they will not agree at Q.

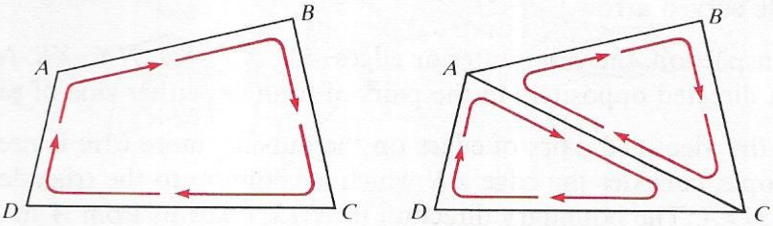
FRAME 7

Orienting each face of a subdivision



FRAME 8

Common edges "directed oppositely"



FRAME 9

Definition: COMPATIBILITY across a common edge

We say that the orientations of two faces having a common edge b are *compatible* across b if the arrows round the boundaries of the two faces are in opposite directions on the two sides of b. (See Frame 8: the common edge $b = AC$ is "directed oppositely" in the two faces.)

BEFORE RESTARTING THE TAPE, READ FRAME 10

FRAME 10

Definition: ORIENTATION of a subdivided surface

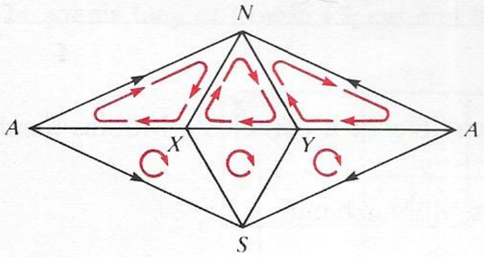
An orientation of a subdivided surface is an orientation of each face (by assigning a direction of travel around its boundary, then shrinking, as in Frame 7) which satisfies the compatibility condition (Frame 9) across each common edge.

RESTART THE TAPE

FRAME 11

Problem

Verify that the sphere is orientable. Use the rule in Frame 10 on the subdivision below; i.e. orient each triangle in turn, checking compatibility as you proceed.

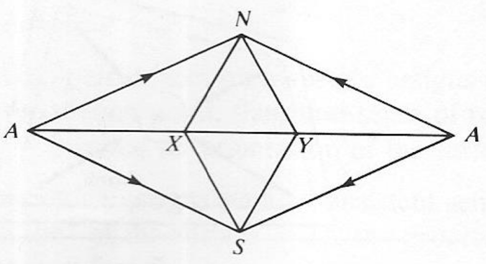


Rhombus representation of the sphere, subdivided into six triangles.

BEFORE RESTARTING THE TAPE, CHECK YOUR ANSWER WITH FRAME 11A

FRAME 11A

Solution for Frame 11



(For simplicity, the orientation in the lower triangles is indicated with a single curved arrow.)

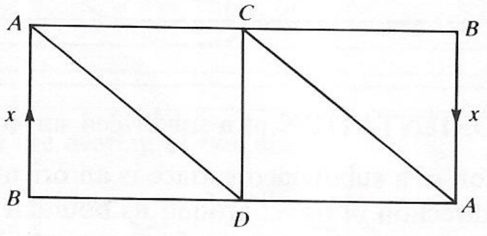
By inspection, the seven interior edges *AX*, *XY*, *YA*, *NX*, *XS*, *NY*, *YS* are each directed oppositely in the pairs of triangles either side of each edge.

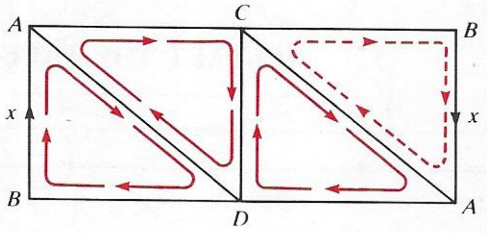
For the identified pairs of edges on the outside, more care is needed. For example, consider the edge *AN* which is common to the triangles *NAX* and *NYA*. The boundary direction in *NAX* takes us from *A* to *N*, but the boundary direction in *NYA* takes us from *N* to *A*. So the edge *AN* is directed oppositely in the two triangles of which it is a common edge. Similarly for the edge *AS*.

RESTART THE TAPE

FRAME 12

Trying to orient a Möbius band





Compatibility condition violated.

FRAME 13

Theorem 1

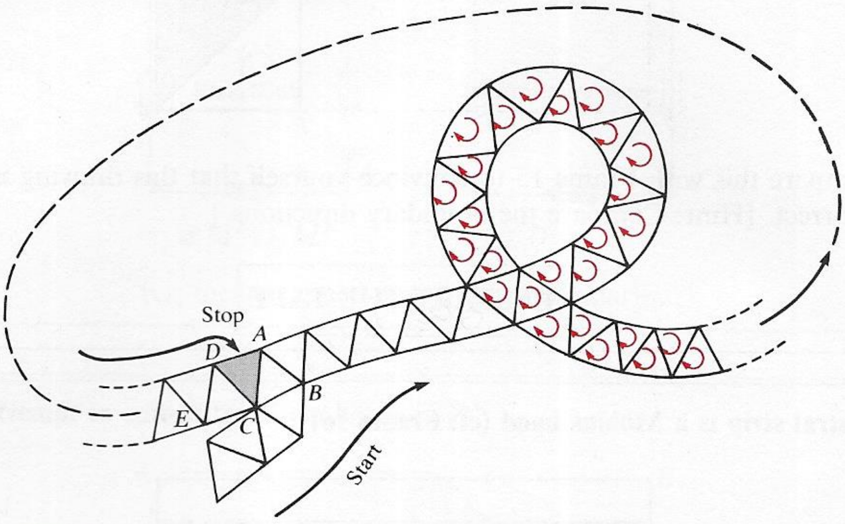
Either a triangulated surface is orientable, or it contains a Möbius band; i.e.

$$\boxed{\begin{array}{c}\text{Contains Möbius band}\\ \Uparrow \\ \text{Non-orientable}\end{array}}$$

FRAME 14

Attempt to orient a non-orientable surface

(Illustrating the proof for Frame 13.)

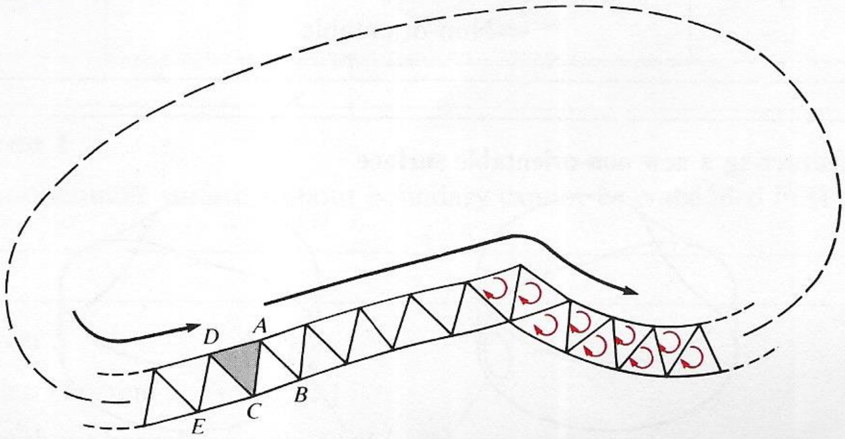


Familiarize yourself with this diagram, then, before restarting the tape, compare it with Frame 15.

FRAME 15

Discarding some triangles in Frame 14

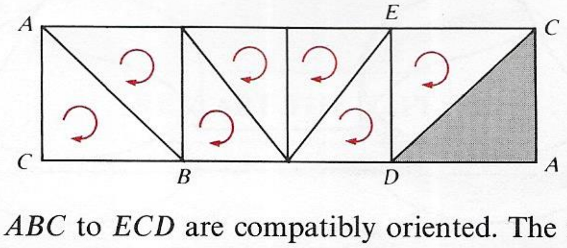
Leaving a simple loop of triangles which is to be cut, later on, along the edge *AC*.



After comparing this with Frame 14, RESTART THE TAPE

FRAME 16

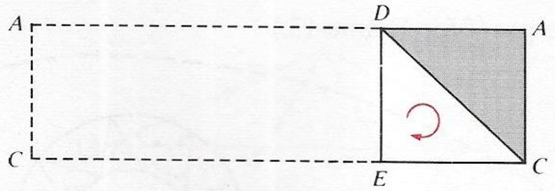
The simple loop of Frame 15, cut and flattened



Triangles from *ABC* to *ECD* are compatibly oriented. The right-hand edge is identified with a half twist to the left-hand edge.

FRAME 17

Different way of drawing the last two triangles

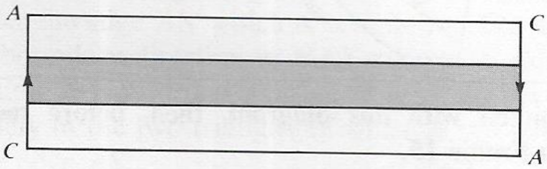


Compare this with Frame 15 to convince yourself that this drawing is incorrect. [Hint: Compare the boundary directions.]

RESTART THE TAPE

FRAME 18

Central strip is a Möbius band (cf. Frame 16)



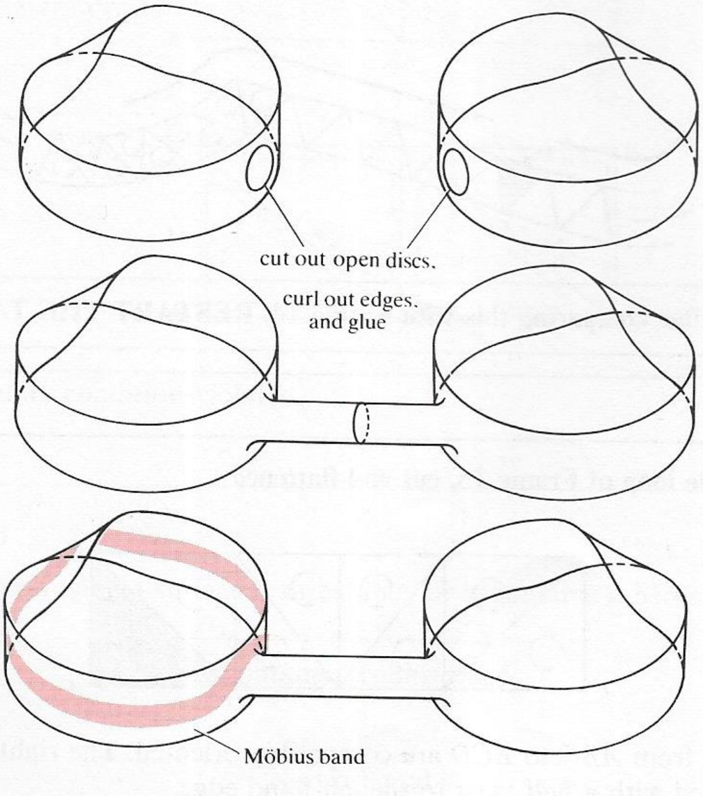
FRAME 19

Theorem 1 of Frame 13 reproduced, plus its converse

$$\begin{tikzcd}
\text{Contains Möbius band} \arrow[d, Rightarrow, shift right=2, "\text{Converse}"'] \\
\text{Non-orientable} \arrow[u, Rightarrow, shift right=2, "\text{Theorem 1}"']
\end{tikzcd}$$

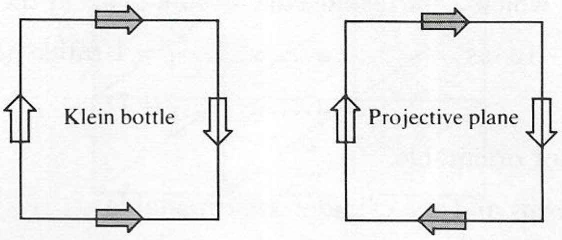
FRAME 20

Constructing a new non-orientable surface


cut out open discs.
curl out edges.
and glue
Möbius band

FRAME 21

Non-orientable surfaces without boundary components

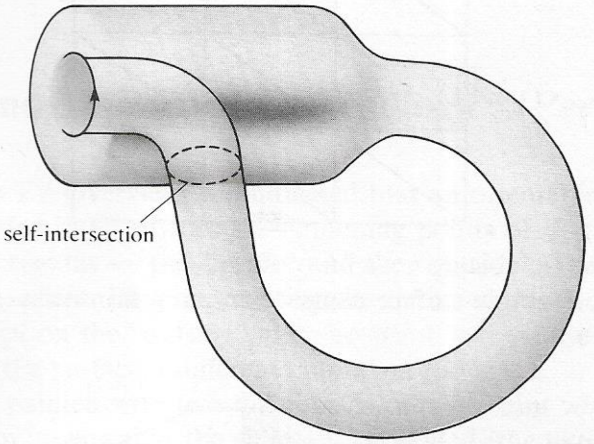


$\cong M^2 \cup_\partial M^2$ $\cong D^2 \cup_\partial M^2$

($\cup_\partial$ means "glued along their boundaries".)

FRAME 22

An attempt to embed the Klein bottle in $\mathbb{R}^3$



FRAME 23

Theorem 2

A non-orientable surface without boundary cannot be embedded in $\mathbb{R}^3$.

FRAME 24

Theorem 3

(Implies Theorem 2, Frame 23.)

S (without boundary) embedded in $\mathbb{R}^3$

$A \Downarrow$

S has an "inside" and an "outside"

$B \Downarrow$

S is orientable

(Hence Theorem 2: S non-orientable, without boundary, cannot be embedded in $\mathbb{R}^3$.)

RESTART THE TAPE

Post-TV Problems

SAQ 15 Suppose a surface S is represented by a polygon with identifications given by an edge equation in which a particular edge occurs twice in the same direction

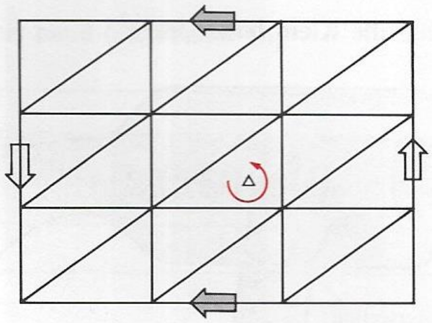
(i.e. as $\ldots a \ldots a \ldots = 1$ rather than

$\ldots a \ldots a^{-1} \ldots = 1$).

Show that S is not orientable.

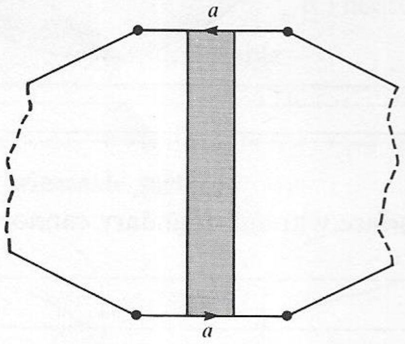
SAQ 16 Verify that the torus and the cylinder are orientable.

SAQ 17 Starting with the following triangulation of the Klein bottle, and the given orientation of the triangle △, orient as many triangles as possible in a compatible way. For one of the non-oriented triangles, T, verify that a chain of oriented triangles can be added to T to form a Möbius band.



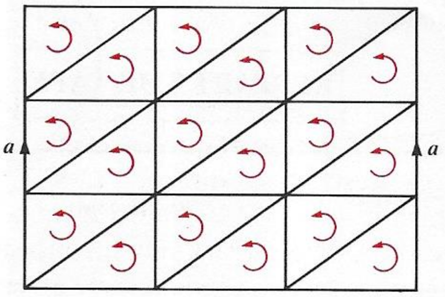
SAQ 18 Prove that an orientable surface cannot contain a non-orientable surface.

SOLUTION TO SAQ 15



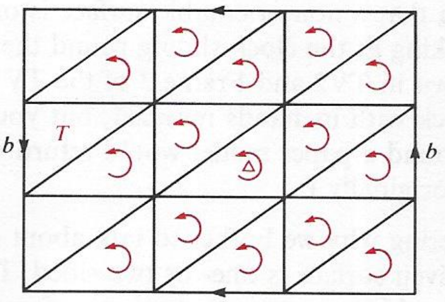
First take a strip joining the middle thirds of the edges a. Then after identifications this becomes a Möbius band contained in S. Hence S is non-orientable. ■

SOLUTION TO SAQ 16 The following is a triangulation and orientation of the cylinder; and, if the top and bottom edges are then identified, of the torus. All that needs to be checked is that the orientations of the triangles are compatible across the boundary edges after identifications, because we started with an orientation of the rectangle.



■

SOLUTION TO SAQ 17



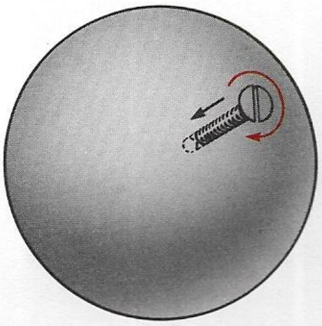
None of the remaining three triangles can be oriented compatibly with the orientations of the other triangles. The triangle T, shown, together with the rest of the middle strip, forms a Möbius band. ■

SOLUTION TO SAQ 18 Suppose S is orientable and contains another surface S'. Then if S' were non-orientable it would contain a Möbius band. This would also lie in S, which would therefore be non-orientable. ■

2.5 One-sided and Two-sided Embeddings (Peripheral)

At the end of the TV Overview, we indicated that an orientable surface without boundary embedded in $\mathbb{R}^3$ divides the remaining points of $\mathbb{R}^3$ (those not on the surface) into two regions — the "inside" and the "outside". The easiest example to think of is the sphere. Imagine painting the sphere with a thick layer of paint. If the brush is kept on the "outside", then no paint reaches the "inside". We could then paint the surface a different colour on the inside, and we would end up with a sphere painted with two different colours of paint which do not meet. We express this by saying that the sphere is two-sided (the two sides corresponding to the two colours).

More precisely, we have said that the sphere is orientable; so we have a coherent sense of clockwise rotation at every point of the sphere. The "right-hand screw" rule then allows us to define a coherent sense of downwards at every point of the sphere.



We can imagine painting the sphere with a layer of paint running "downwards" from the brush towards the surface, thus defining one side of the surface. The opposite direction to downwards is defined to be upwards, and in the same way this defines the other side. A similar argument would apply to any orientable surface, and could be used to prove that an orientable surface without boundary in $\mathbb{R}^3$ has two sides.

For a non-orientable surface there is, by definition, no coherent sense of clockwise rotation and, therefore, no coherent sense of downwards. Therefore we cannot decide unambiguously which direction away from the surface at a point P is upwards and which is downwards. Thus we cannot see how to paint just one side of the surface at a time. In fact, any attempt to do this will eventually involve painting the surface from both directions at each point. This is often expressed by saying that a non-orientable surface is one-sided. (This gives another way of looking at the clock sliding round the centre circle of a Möbius band, which is shown in TV2 and Frame 3 of the TV Overview. After the trip, the clock comes back with its hands reversed; but you may have noticed that a real clock sliding round a paper model would return with its back facing the way the dial faced originally.)

You may be wondering why we bother to talk about orientability rather than simply whether a given surface is one- or two-sided. The full answer is rather complicated, but the chief reason is that some surfaces (like the projective plane and the Klein bottle) cannot be embedded in $\mathbb{R}^3$ anyway (models with self-intersections do not count). Further, whether or not a particular representation of a surface in another space is one- or two-sided depends both on the representation and on the space. The details would be out of place in this course, but **B** gives an example on page 195 of a space (manifold) in which the Klein bottle can be either one-sided or two-sided. In the same space, the torus can be one-sided or two-sided, depending on which representation is chosen. But orientability is not subject to this ambiguity — it is an *intrinsic property of the surface itself*. In the example just quoted, the Klein bottle is still non-orientable, and the torus orientable. In *Unit 3* you will see that there is an analogue for curves lying in a surface. A curve may be two-sided (like a circle in $\mathbb{R}^2$) or one-sided (like the central circle of a Möbius band).

CHALLENGE PROBLEM

Suppose we have a non-orientable surface in $\mathbb{R}^3$.

(i) Prove that there is an orientation-reversing path which starts and finishes at any desired point of the surface. (An orientation-reversing path is one such that a clockface sliding around it will return with its hands reversed.)

(ii) Deduce that if you start painting the surface with a downward-pointing brush at any point, then you can paint a line which returns to that point with the brush pointing upwards. (Hence the surface is one-sided.)

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